Lecture 11

Dynamic Bayesian Networks and Hidden Markov Models Decision Trees

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Course Overview

- ✔ Introduction
  - ✔ Artificial Intelligence
  - ✔ Intelligent Agents
- ✔ Search
  - ✔ Uninformed Search
  - ✔ Heuristic Search
- ✔ Adversarial Search
  - ✔ Minimax search
  - ✔ Alpha-beta pruning
- ✔ Knowledge representation and Reasoning
  - ✔ Propositional logic
  - ✔ First order logic
  - ✔ Inference
- ✔ Uncertain knowledge and Reasoning
  - ✔ Probability and Bayesian approach
  - ✔ Bayesian Networks
  - Hidden Markov Chains
  - Kalman Filters
- Learning
  - Decision Trees
  - Maximum Likelihood
  - EM Algorithm
  - Learning Bayesian Networks
  - Neural Networks
  - Support vector machines

Performance of approximation algorithms

- Absolute approximation: $|P(X|\mathbf{e}) - \hat{P}(X|\mathbf{e})| \leq \epsilon$

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- Polytime approximation: $\text{poly}(n, \epsilon^{-1}, \log \delta^{-1})$
- Theorem (Dagum and Luby, 1993): both absolute and relative approximation for either deterministic or randomized algorithms are NP-hard for any $\epsilon, \delta < 0.5$
(Absolute approximation polytime with no evidence—Chernoff bounds)

Summary

Exact inference by variable elimination:

- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

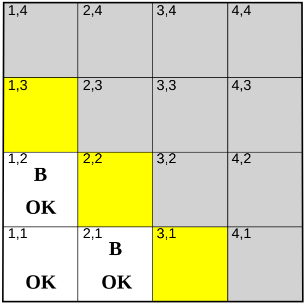
Approximate inference by Likelihood Weighting (LW), Markov Chain Monte Carlo Method (MCMC):

- PriorSampling and RejectionSampling unusable as evidence grow
  - LW does poorly when there is lots of (late-in-the-order) evidence
  - LW, MCMC generally insensitive to topology
  - Convergence can be very slow with probabilities close to 1 or 0
  - Can handle arbitrary combinations of discrete and continuous variables

Outline

1. Exercise
2. Uncertainty over Time
3. Speech Recognition
4. Learning

Wumpus World



$P_{ij} = true$ iff $[i, j]$ contains a pit
$B_{ij} = true$ iff $[i, j]$ is breezy
Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model

Specifying the probability model

The full joint distribution is $\mathbf{P}(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$

Apply product rule: $\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\mathbf{P}(P_{1,1}, \ldots, P_{4,4})$

(Do it this way to get $P(\mathit{Effect}|\mathit{Cause})$.)

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(Do it this way to get $P(Effect|Cause)$.)

First term: 1 if pits are adjacent to breezes, 0 otherwise

Second term: pits are placed randomly, probability 0.2 per square:

$$\mathbf{P}(P_{1,1}, \ldots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

Observations and query

We know the following facts:

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

Query is $\mathbf{P}(P_{1,3} | known, b)$

Define $Unknown$ = P_{ij}s other than $P_{1,3}$ and $Known$

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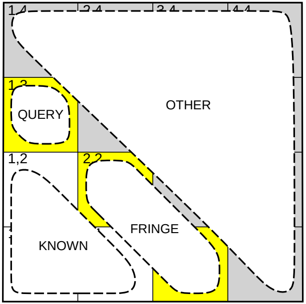
For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

Grows exponentially with number of squares!

Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



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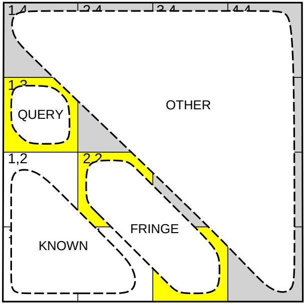

Define $Unknown = Fringe \cup Other$
$\mathbf{P}(b|P_{1,3}, Known, Unknown) = \mathbf{P}(b|P_{1,3}, Known, Fringe)$
Manipulate query into a form where we can use this!

Using conditional independence contd.

$$\mathbf{P}(P_{1,3}|known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

Using conditional independence contd.

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$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|known, P_{1,3}, fringe, other)\mathbf{P}(P_{1,3}, known, fringe, other)$$

Using conditional independence contd.

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\mathbf{P}(P_{1,3}|known, b) &= \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b) \\
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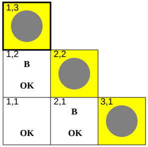
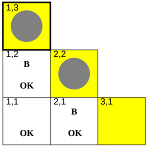
Using conditional independence contd.



0.2 x 0.2 = 0.04



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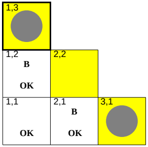

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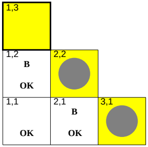

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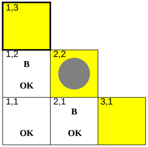

0.2 x 0.8 = 0.16

$$\begin{aligned}\mathbf{P}(P_{1,3}|known, b) &= \alpha' \langle 0.2(0.04+0.16+0.16),\ 0.8(0.04+0.16) \rangle \\ &\approx \langle 0.31, 0.69 \rangle\end{aligned}$$

$$\mathbf{P}(P_{2,2}|known, b) \approx \langle 0.86, 0.14 \rangle$$

Outline

1. Exercise
2. Uncertainty over Time
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Outline

- Time and uncertainty
- Inference: filtering, prediction, smoothing
- Hidden Markov models
- Kalman filters (a brief mention)
- Dynamic Bayesian networks (an even briefer mention)

Time and uncertainty

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  $\mathbf{X}_t$ = set of unobservable state variables at time t
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- Notation: $\mathbf{X}_{a:b} = \mathbf{X}_a, \mathbf{X}_{a+1}, \ldots, \mathbf{X}_{b-1}, \mathbf{X}_b$

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Markov assumption: $\mathbf{X}_t$ depends on **bounded** subset of $\mathbf{X}_{0:t-1}$
First-order Markov process: $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
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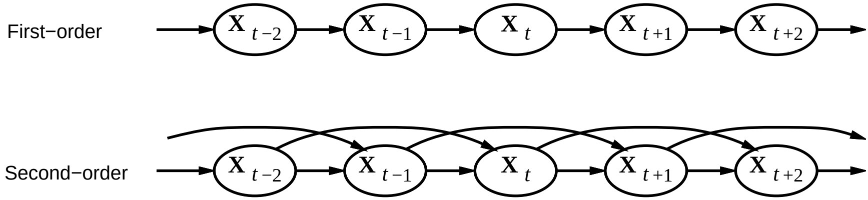

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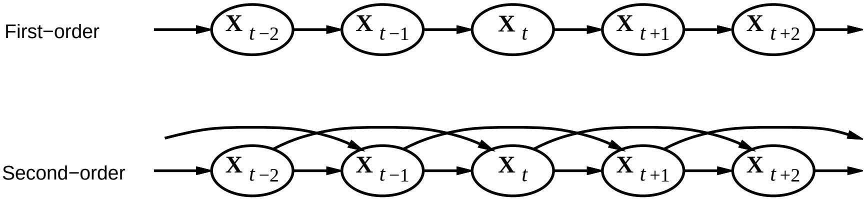

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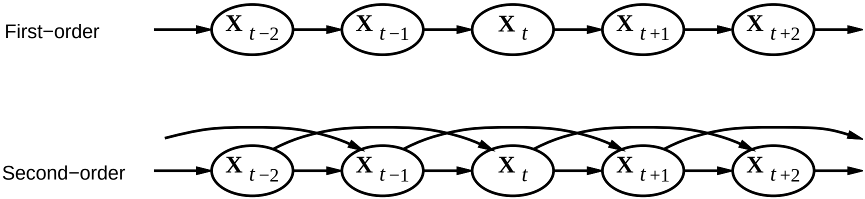
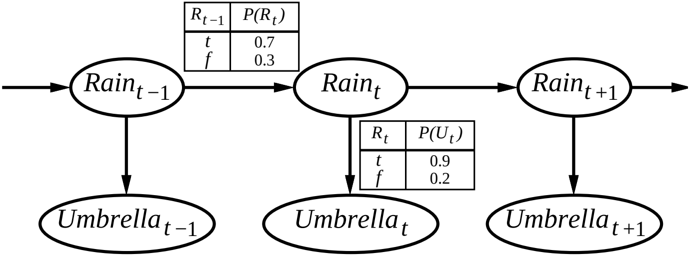


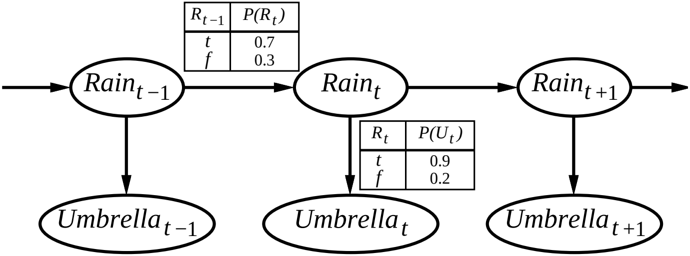
Sensor Markov assumption: $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$
$\rightsquigarrow$ Stationary process:

- transition model $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$ and
- sensor model $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$ fixed for all t

Example


R_{t-1} | $P(R_t)$
t 0.7
f 0.3
$Rain_{t-1}$
$Rain_t$
$Rain_{t+1}$
R_t | $P(U_t)$
t 0.9
f 0.2
$Umbrella_{t-1}$
$Umbrella_t$
$Umbrella_{t+1}$

Example



First-order Markov assumption not exactly true in real world!
Possible fixes:

1. **Increase order** of Markov process
2. **Augment state**, e.g., add $Temp_t$, $Pressure_t$

Example: robot motion.
Augment position and velocity with $Battery_t$

Inference tasks

1. Filtering: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
   belief state—input to the decision process of a rational agent

2. Prediction: $\mathbf{P}(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$ for $k > 0$
   evaluation of possible action sequences;
   like filtering without the evidence

3. Smoothing: $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$
   better estimate of past states, essential for learning

4. Most likely explanation: $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$
   speech recognition, decoding with a noisy channel

Filtering

Aim: devise a **recursive** state estimation algorithm:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}))$$

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I.e., prediction + estimation.

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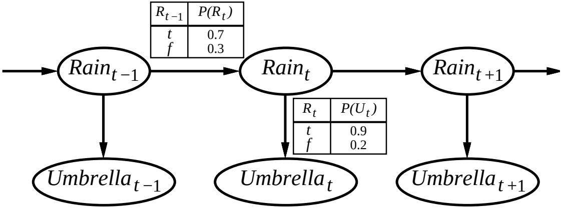
I.e., prediction + estimation. Prediction by summing out $\mathbf{X}_t$:

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t})P(\mathbf{x}_t|\mathbf{e}_{1:t}) \\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t})\end{aligned}$$

$\mathbf{f}_{1:t+1} = \text{Forward}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$ where $\mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
Time and space **constant** (independent of t) by keeping track of $\mathbf{f}$

Filtering example



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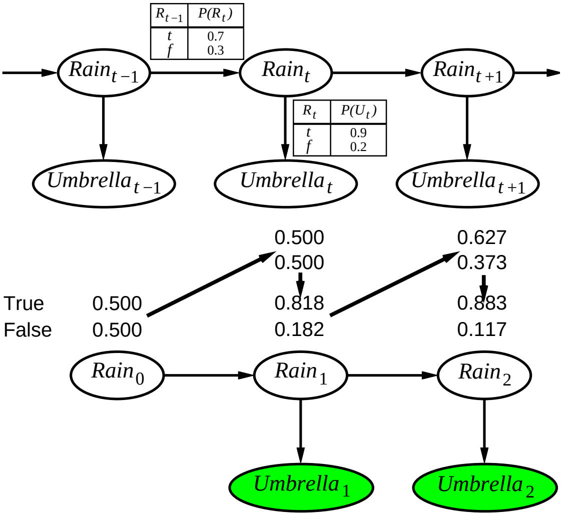
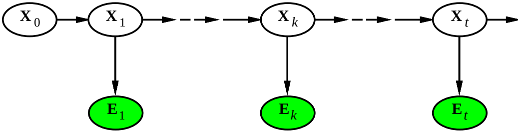


Smoothing



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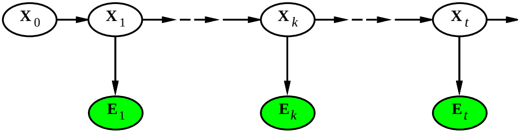

Divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$, $\mathbf{e}_{k+1:t}$:

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
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&= \alpha\mathbf{f}_{1:k}\mathbf{b}_{k+1:t}
\end{aligned}$$

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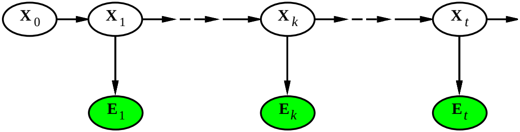

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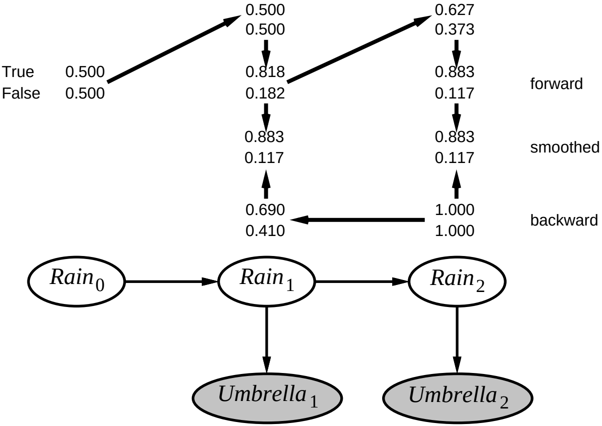
\end{aligned}
$$

Backward message computed by a backwards recursion:

$$
\begin{aligned}
\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)
\end{aligned}
$$

Smoothing example



If we want to smooth the whole sequence:
Forward–backward algorithm: cache forward messages along the way
Time linear in t (polytree inference), space $O(t|\mathbf{f}|)$

Most likely explanation

Most likely sequence $\neq$ sequence of most likely states (joint distr.)!

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= most likely path to **some** $\mathbf{x}_t$ plus one more step

$$\max_{\mathbf{x_1}\ldots\mathbf{x_t}} \mathbf{P}(\mathbf{x}_1,\ldots,\mathbf{x}_t,\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$$

$$= \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x_t}} \left(\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) \max_{\mathbf{x_1}\ldots\mathbf{x_{t-1}}} P(\mathbf{x}_1,\ldots,\mathbf{x}_{t-1},\mathbf{x}_t|\mathbf{e}_{1:t})\right)$$

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$$\max_{\mathbf{x}_1 \ldots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$$
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Identical to filtering, except $\mathbf{f}_{1:t}$ replaced by

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t}),$$

I.e., $\mathbf{m}_{1:t}(i)$ gives the probability of the most likely path to state i.

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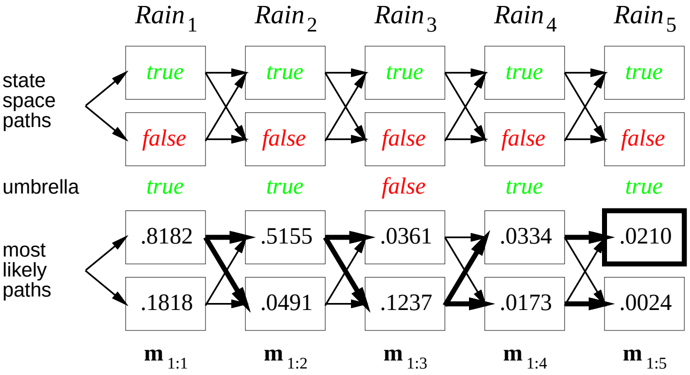
Update has sum replaced by max, giving the Viterbi algorithm:

$$\mathbf{m}_{1:t+1} = \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x_t}} (\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)\mathbf{m}_{1:t})$$

Viterbi example



Hidden Markov models

$\mathbf{X}_t$ is a single, discrete variable (usually $\mathbf{E}_t$ is too)
Domain of X_t is $\{1, \ldots, S\}$
Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$, e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$
Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
e.g., with $U_1 = true$, $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$
Forward and backward messages as column vectors:

$$\begin{aligned} \mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^{\top} \mathbf{f}_{1:t} \\ \mathbf{b}_{k+1:t} &= \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t} \end{aligned}$$

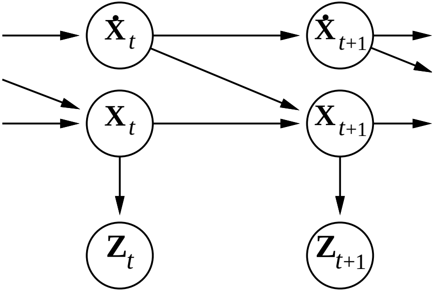
Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$

Kalman filters

Modelling systems described by a set of continuous variables,

e.g., tracking a bird flying—$\mathbf{X}_t = X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$.

Airplanes, robots, ecosystems, economies, chemical plants, planets, ...



Gaussian prior, linear Gaussian transition model and sensor model

Updating Gaussian distributions

Prediction step: if $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$ is Gaussian, then prediction

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \int_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t})\, d\mathbf{x}_t$$

is Gaussian. If $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$ is Gaussian, then the updated distribution

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

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Hence $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$ is multivariate Gaussian $N(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ for all t

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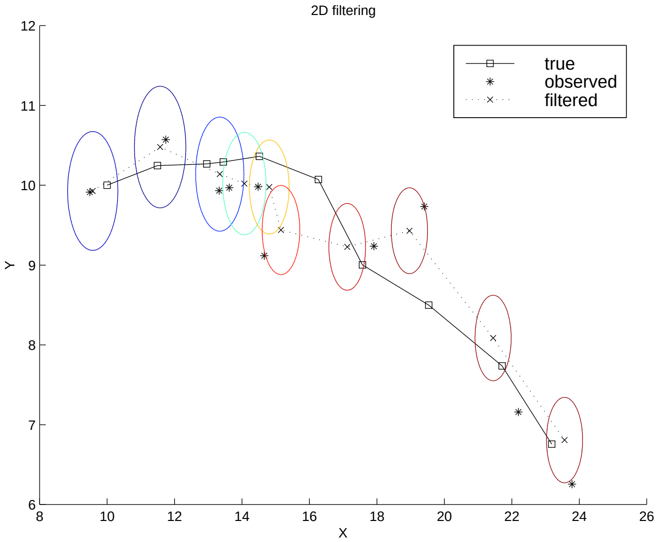
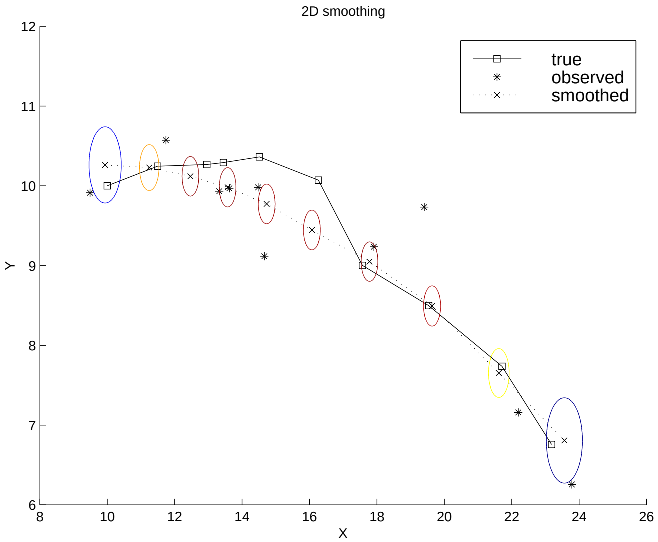
General (nonlinear, non-Gaussian) process: description of posterior grows **unboundedly** as $t \to \infty$

2-D tracking example: filtering

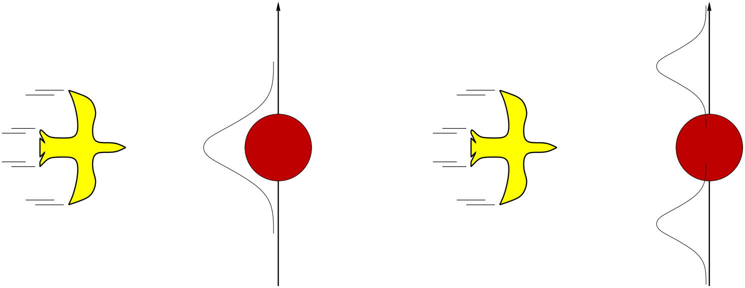


2-D tracking example: smoothing



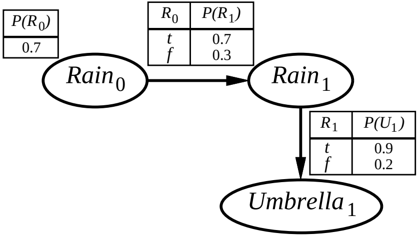
Where it breaks

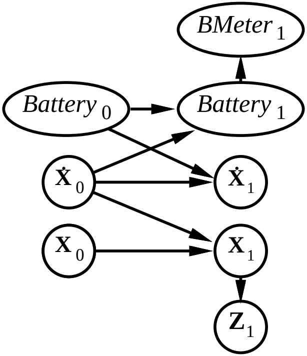
Cannot be applied if the transition model is nonlinear
Extended Kalman Filter models transition as **locally linear** around $\mathbf{x}_t = \mu_t$
Fails if systems is locally unsmooth



Dynamic Bayesian networks

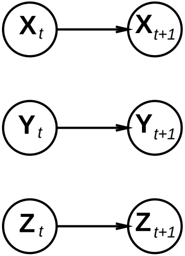
$\mathbf{X}_t$, $\mathbf{E}_t$ contain arbitrarily many variables in a replicated Bayes net

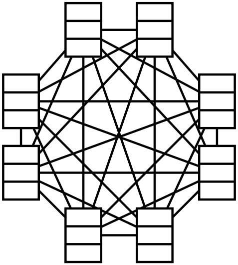




DBNs vs. HMMs

Every HMM is a single-variable DBN; every discrete DBN is an HMM





Sparse dependencies $\Rightarrow$ exponentially fewer parameters;
e.g., 20 state variables, three parents each
DBN has $20 \times 2^3 = 160$ parameters, HMM has $2^{20} \times 2^{20} \approx 10^{12}$

DBNs vs Kalman filters

Every Kalman filter model is a DBN, but few DBNs are KFs;
real world requires non-Gaussian posteriors

Summary

- Temporal models use state and sensor variables replicated over time
- Markov assumptions and stationarity assumption, so we need
  – transition model $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
  – sensor model $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$
- Tasks are filtering, prediction, smoothing, most likely sequence; **all done recursively with constant cost per time step**
- Hidden Markov models have a single discrete state variable; used for speech recognition
- Kalman filters allow n state variables, linear Gaussian, $O(n^3)$ update
- Dynamic Bayes nets subsume HMMs, Kalman filters; exact update intractable

Outline

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- Speech as probabilistic inference
- Speech sounds
- Word pronunciation
- Word sequences

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- Speech signals are noisy, variable, ambiguous

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  I.e., decomposes into acoustic model + language model
- *Words* are the hidden state sequence, *signal* is the observation sequence

Phones

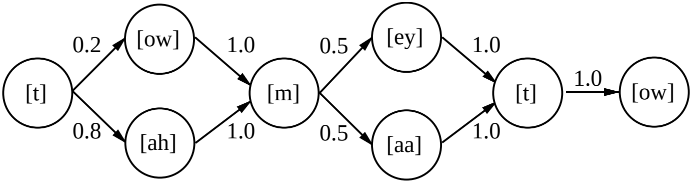
All human speech is composed from 40-50 phones, determined by the configuration of articulators (lips, teeth, tongue, vocal cords, air flow)
Form an intermediate level of hidden states between words and signal
$\Rightarrow$ acoustic model = pronunciation model + phone model
ARPAbet designed for American English

[iy]	b**ea**t	[b]	**b**et	[p]	**p**et
[ih]	b**i**t	[ch]	**Ch**et	[r]	**r**at
[ey]	b**e**t	[d]	**d**ebt	[s]	**s**et
[ao]	b**ough**t	[hh]	**h**at	[th]	**th**ick
[ow]	b**oa**t	[hv]	**h**igh	[dh]	**th**at
[er]	B**er**t	[l]	**l**et	[w]	**w**et
[ix]	ros**e**s	[ng]	si**ng**	[en]	butt**on**
⋮	⋮	⋮	⋮	⋮	⋮

E.g., "ceiling" is [s iy l ih ng] / [s iy l ix ng] / [s iy l en]

Word pronunciation models

Each word is described as a distribution over phone sequences
Distribution represented as an HMM transition model



$$P([towmeytow]|\text{“tomato”}) = P([towmaatow]|\text{“tomato”}) = 0.1$$
$$P([tahmeytow]|\text{“tomato”}) = P([tahmaatow]|\text{“tomato”}) = 0.4$$

Structure is created manually, transition probabilities learned from data

Isolated words

- Phone models + word models fix likelihood $P(e_{1:t}|word)$ for isolated word

$$P(word|e_{1:t}) = \alpha P(e_{1:t}|word)P(word)$$

- Prior probability $P(word)$ obtained simply by counting word frequencies $P(e_{1:t}|word)$ can be computed recursively: define

$$\boldsymbol{\ell}_{1:t} = \mathbf{P}(\mathbf{X}_t, \mathbf{e}_{1:t})$$

  and use the recursive update

$$\boldsymbol{\ell}_{1:t+1} = \text{Forward}(\ell_{1:t}, \mathbf{e}_{t+1})$$

  and then $P(e_{1:t}|word) = \sum_{\mathbf{x_t}} \boldsymbol{\ell}_{1:t}(\mathbf{x}_t)$

- Isolated-word dictation systems with training reach 95–99% accuracy

Continuous speech

Not just a sequence of isolated-word recognition problems!

- Adjacent words highly correlated
- Sequence of most likely words $\neq$ most likely sequence of words
- Segmentation: there are few gaps in speech
- Cross-word coarticulation—e.g., “next thing”

Continuous speech systems manage 60–80% accuracy on a good day

Language model

Prior probability of a word sequence is given by chain rule:

$$P(w_1 \cdots w_n) = \prod_{i=1}^{n} P(w_i | w_1 \cdots w_{i-1})$$

Bigram model:

$$P(w_i | w_1 \cdots w_{i-1}) \approx P(w_i | w_{i-1})$$

Train by counting all word pairs in a large text corpus
More sophisticated models (trigrams, grammars, etc.) help a little bit

Combined HMM

- States of the combined language+word+phone model are labelled by the word we're in + the phone in that word + the phone state in that phone
- Viterbi algorithm finds the most likely **phone state** sequence
- Does segmentation by considering all possible word sequences and boundaries
- Doesn't always give the most likely word sequence because each word sequence is the sum over many state sequences
- Jelinek invented A* in 1969 a way to find most likely word sequence where "step cost" is $-\log P(w_i|w_{i-1})$

Outline

Outline

- Learning agents
- Inductive learning
- Decision tree learning
- Measuring learning performance

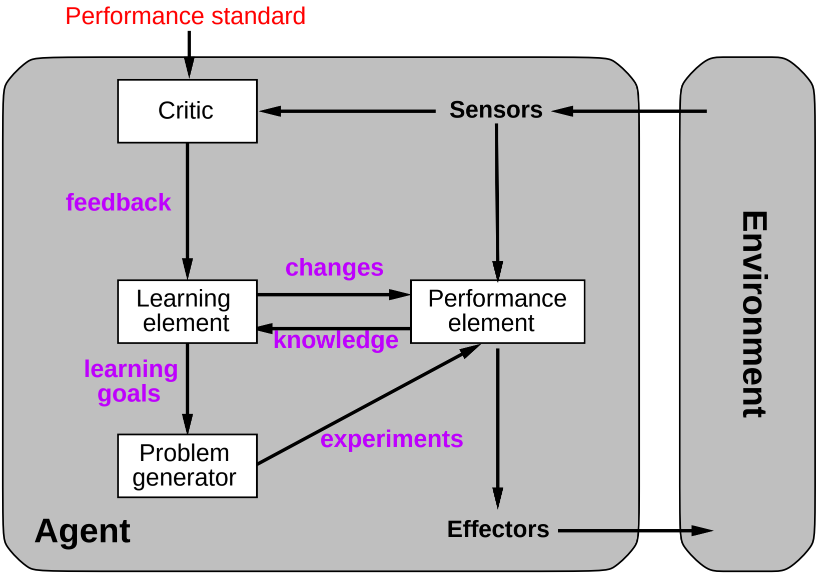
Learning

Back to Turing's article:
- child mind program
- education

Reward & Punishment

- Learning is essential for unknown environments,
  i.e., when designer lacks omniscience
- Learning is useful as a system construction method,
  i.e., expose the agent to reality rather than trying to write it down
- Learning modifies the agent's decision mechanisms to improve performance

Learning agents



Learning element

Design of learning element is dictated by

- ♢ what type of performance element is used
- ♢ which functional component is to be learned
- ♢ how that functional compoent is represented
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Example scenarios:

Performance element	Component	Representation	Feedback
Alpha−beta search	Eval. fn.	Weighted linear function	Win/loss
Logical agent	Transition model	Successor−state axioms	Outcome
Utility−based agent	Transition model	Dynamic Bayes net	Outcome
Simple reflex agent	Percept−action fn	Neural net	Correct action

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Supervised learning: correct answers for each instance
Reinforcement learning: occasional rewards