DM877

Constraint Programming

Set Variables

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Resume and Outlook

- Modeling in CP
- Global constraints (declaration)
- Notions of local consistency
- Global constraints (operational: filtering algorithms)
- Search
- Set variables
- Symmetry breaking

Global Variables

Global variables: complex variable types representing combinatorial structures in which problems find their most natural formulation

Eg:
sets, multisets, strings, functions, graphs
bin packing, set partitioning, mapping problems

We will see:

- Set variables

- Graph variables

Outline

1. Set Variables

2. Graph Variables

3. Float Variables

Finite-Set Variables

- A finite-domain integer variable takes values from a finite set of integers.
- A finite-domain set variable takes values from the power set of a finite set of integers.
  Eg.:
  domain of x is the set of subsets of $\{1, 2, 3\}$:

$$\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Finite-Set Variables

Recall the shift-assignment problem

We have a lower and an upper bound on the number of shifts that each worker is to staff (symmetric cardinality constraint)

- one variable for each worker that takes as value the set of shifts covererd by the worker. $\rightsquigarrow$ exponential number of values
- set variables with domain $D(x) = [lb(x), ub(x)]$
  $D(x)$ represented by two sets:
  - $lb(x)$ mandatory elements
  - $ub(x) \setminus lb(x)$ of possible elements

  The value assigned to x should be a set $s(x)$ such that $lb(x) \subseteq s(x) \subseteq ub(x)$

In practice good to keep dual views with channelling

Finite-Set Variables

Example:

domain of x is the set of subsets of $\{1, 2, 3\}$:

$$\{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

can be represented in space-efficient way by:

$$[\{\}..\{1, 2, 3\}]$$

The representation is however an approximation!

Example:

domain of x is the set of subsets of $\{1, 2, 3\}$:

$$\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

cannot be captured exactly by an interval. The closest interval would be still:

$$[\{\}..\{1, 2, 3\}]$$

Set Variables

Definition

set variable is a variable with domain $D(x) = [lb(x), ub(x)]$
$D(x)$ represented by two sets:

- $lb(x)$ mandatory elements (intersection of all subsets)
- $ub(x) \setminus lb(x)$ of possible elements (union of all subsets)

The value assigned to x must be a set $s(x)$ such that $lb(x) \subseteq s(x) \subseteq ub(x)$

We are not interested in domain consistency but in bound consistency:

Enforcing bound consistency

A bound consistency for a constraint C defined on a set variable x requires that we:

- Remove a value v from $ub(x)$ if there is no solution to C in which $v \in s(x)$.
- Include a value $v \in ub(x)$ in $lb(x)$ if in all solutions to C, $v \in s(x)$.

In Gecode

```
#include <gecode/set.hh>
SetVar(Space home, int glbMin, int glbMax, int lubMin, int lubMax, int
cardMin=MIN, int cardMax=MAX); // greatest lower bound; least upper bound
```

```
SetVar A(home, 0, 1, 0, 5, 3, 3);
cout << A: {0,1}..{0..5}#(3) // prints a set variable
```

```
A.glbSize();  2 // num. of elements in the greatest lower bound
A.glbMin();  0 // minimum element of greatest lower bound
A.glbMax();  1 // maximum of greatest lower bound
for (SetVarGlbValues i(x); i(); ++i) cout << i.val() << ' '; // values of glb
for (SetVarGlbRanges i(x); i(); ++i) cout << i.min() << ".." << i.max();

A.lubSize():  6 // num. of elements in the least upper bound
A.lubMin():  0 // minimum element of least upper bound
A.lubMax():  5 // maximum element of least upper bound
for (SetVarLubValues i(x); i(); ++i) cout << i.val() << ' ';
for (SetVarLubRanges i(x); i(); ++i) cout << i.min() << ".." << i.max();

A.unknownSize():  4 // num. of unknown elements (elements in lub but not in glb)
for (SetVarUnknownValues i(x); i(); ++i) cout << i.val() << ' ';
for (SetVarUnknownRanges i(x); i(); ++i) cout << i.min() << ".." <<i.max();

A.cardMin():  3 // cardinality minimum
A.cardMax():  3 // cardinality maximum
```

In Gecode

```
SetVar(home, IntSet glb, int lubMin, int lubMax, int cardMin=MIN, int cardMax=MAX)
```

```
SetVar A(home, IntSet(), 0, 5, 0, 4)
```

```
cout << A;
A.glbSize():  0 // num. of elements in the greatest lower bound
A.glbMin():  -1073741823 // minimum element of greatest lower bound
A.glbMax():  1073741823 // maximum of greatest lower bound

A.lubSize():  6 // num. of elements in the least upper bound
A.lubMin():  0 // minimum element of least upper bound
A.lubMax():  5 // maximum element of least upper bound

A.unknownSize)():  6 // num. of unknown elements (elements in lub but not in glb)

A.cardMin():  0 // cardinality minimum
A.cardMax():  4 // cardinality maximum
```

In Gecode

```
SetVar(home, int glbMin, int glbMax, IntSet lub, int cardMin=MIN, int cardMax=MAX)
```

```
SetVar A(home, 1, 3, IntSet({ {1,4}, {8,12} }), 2, 4)
```

```
cout << A;
A.glbSize(A):  3 // num. of elements in the greatest lower bound
A.glbMin(A):  1 // minimum element of greatest lower bound
A.glbMax(A):  3 // maximum of greatest lower bound

A.lubSize(A):  9 // nuA. of elements in the least upper bound
A.lubMin(A):  1 // minimum element of least upper bound
A.lubMax(A):  12 // maximum element of least upper bound


// A.unknownValues(A):  [4, 8, 9, 10, 11, 12]
A.unknownSize)(A):  6 // num. of unknown elements (elements in lub but not in glb)
// A.unknownRanges(A):  [(4, 4), (8, 12)]

A.cardMin(A):  3 // cardinality minimum
A.cardMax(A):  4 // cardinality maximum
```

Social Golfers Problem

Find a schedule for a golf tournament:

- $g \cdot s$ golfers,
- who want to play a tournament in g groups of s golfers over w weeks,
- such that no two golfers play against each other more than once during the tournament.

A solution for the instance $w = 4, g = 3, s = 3$
(players are numbered from 0 to 8)

	Group 0			*Group 1*			*Group 2*		
Week 0	0	1	2	3	4	5	6	7	8
Week 1	0	3	6	1	4	7	2	5	8
Week 2	0	4	8	1	5	6	2	3	7
Week 3	0	5	7	1	3	8	2	4	6

Constraint Programming

Model with Integer Variables

```
players = 9;
groupSize = 3;
days = 4;

groups = players/groupSize;

#=== Variables ===============
assign = m.intvars(players * days, 0, groups-1)
schedule = Matrix(players, days, assign)

#=== Constraints =============
# C1: Each group has exactly groupSize players
for d in range(days):
    m.count(schedule.col(d), [groupSize, groupSize, groupSize]);

# C2: Each pair of players only meets once
p_pairs = [(a,b) for a in range(players) for b in range(players) if p1<p2]
d_pairs = [(a,b) for a in range(days) for b in range(days) if d1<d2]
for (p1,p2) in p_pairs:
    for (d1,d2) in d_pairs:
        b1 = m.boolvar()
        b2 = m.boolvar()
        m.rel(assign(p1,d1), IRT_EQ, assign(p2,d1), b1)
        m.rel(assign(p1,d2), IRT_EQ, assign(p2,d2), b2)
        m.linear([b1,b2], IRT_LQ, 1)

m.branch(assign, INT_VAL_MIN_MIN, INT_VAL_SPLIT_MIN)
```

Model with Finite Set Variables

Array of set variables:

```
int w = 4;
int g = 3;
int s = 3;

int golfers = g * s;

SetVarArray groups(home, w*g, IntSet(), 0, golfers-1, s, s)
```

size $g \cdot w$, where each group can contain the players $[0..g \cdot s - 1]$ and has cardinality s

```
array[WEEK,GROUP] of var set of GOLFER: Sched; % In Minizinc
```

Constraints on FS variables
Domain constraints

```
dom(home, x, SRT_SUB, 1, 10);
dom(home, x, SRT_SUP, 1, 3);
dom(home, y, SRT_DISJ, IntSet(4, 6));
```

```
cardinality(home, x, 3, 5);
```

In MiniZinc:
the number of elements in the set `card`.

Constraints on FS variables
Relation constraints

```
rel(home, x, IRT_LE, y)
```

```
rel(home, x, SRT_SUB, y)
```

In MiniZinc:

- `in` element membership
- `subset` (non-strict) subset relationship
- `superset` (non-strict) superset relationship

Constraints on FS variables

Set operations

```
rel(x, SOT_UNION, y, SRT_EQ, z)
```

```
rel(SOT_UNION, x, y)
```

In MiniZinc: standard set operations are provided:

- `union` union
- `intersect` intersection
- `diff` set difference
- `symdiff` symmetric set difference

Constraints on FS variables

Element

```
element(home, x, y, z)
```

for an array of set variables or constants x,
an integer variable y,
and a set variable z.

It enforces z to be the element of array x at index y (where the index starts at 0).

Example

```
element([{{1,2,3},{2,3},{3,4}},{{2,3},{2}},{{1,4},{3,4},{3}}], 3, z)
```

=> z={{1,4},{3,4},{3}}

Constraints on FS variables

Set Global Cardinality

it bounds the minimum and maximum number of occurrences of an element in an array of set variables:

$$\forall v \in U : l_v \leq |\mathcal{S}_v| \leq u_v$$

where $\mathcal{S}_v$ is the set of set variables that contain the element v, i.e., $\mathcal{S}_v = \{s \in S : v \in s\}$

(not present in gecode)

Constraints on FS variables
Set Global Cardinality

Bessiere et al. [2004]

Table 1. Intersection × Cardinality.

	$\forall i < j \ \ldots$			
$\forall k \ \ldots$	$\lvert X_i \cap X_j \rvert = 0$	$\lvert X_i \cap X_j \rvert \leq k$	$\lvert X_i \cap X_j \rvert \geq k$	$\lvert X_i \cap X_j \rvert = k$
-	`Disjoint` polynomial *decomposable*	$\texttt{Intersect}_{\leq}$ polynomial *decomposable*	$\texttt{Intersect}_{\geq}$ polynomial *decomposable*	$\texttt{Intersect}_{=}$ NP-hard *not decomposable*
$\lvert X_k \rvert > 0$	`NEDisjoint` polynomial *not decomposable*	$\texttt{NEIntersect}_{\leq}$ polynomial *decomposable*	$\texttt{NEIntersect}_{\geq}$ polynomial *decomposable*	$\texttt{FCIntersect}_{=}$ NP-hard *not decomposable*
$\lvert X_k \rvert = m_k$	`FCDisjoint` poly on sets, NP-hard on multisets *not decomposable*	$\texttt{FCIntersect}_{\leq}$ NP-hard *not decomposable*	$\texttt{FCIntersect}_{\geq}$ NP-hard *not decomposable*	$\texttt{NEIntersect}_{=}$ NP-hard *not decomposable*

Table 2. Partition + Intersection × Cardinality.

	$\bigcup_i X_i = X \ \wedge \ \forall i < j \ \ldots$			
$\forall k \ \ldots$	$\lvert X_i \cap X_j \rvert = 0$	$\lvert X_i \cap X_j \rvert \leq k$	$\lvert X_i \cap X_j \rvert \geq k$	$\lvert X_i \cap X_j \rvert = k$
-	`Partition`: polynomial *decomposable*	?	?	?
$\lvert X_k \rvert > 0$	`NEPartition`: polynomial *not decomposable*	?	?	?
$\lvert X_k \rvert = m_k$	`FCPartition` polynomial on sets, NP-hard on multisets *not decomposable*	?	?	?

Constraints on FS variables

Constraints connecting set and integer variables

the integer variable y is equal to the cardinality of the set variable x.

```
cardinality(home, x, y);
```

Minimal and maximal elements of a set: int var y is minimum of set var x

```
min(x, y);
```

Weighted sets: assigns a weight to each possible element of a set variable x, and then constrains an integer variable y to be the sum of the weights of the elements of x

```
int e[6] = {1, 3, 4, 5, 7, 9};
int w[6] = {-1, 4, 1, 1, 3, 3}
weights(home, e, w, x, y)
```

enforces that x is a subset of $\{1, 3, 4, 5, 7, 9\}$ (the set of elements), and that y is the sum of the weights of the elements in x, where the weight of the element 1 would be -1, the weight of 3 would be 4 and so on.
Eg. Assigning x to the set $\{3, 7, 9\}$ would therefore result in y be set to $4 + 3 + 3 = 10$

Constraints on FS variables
Channeling constraints

an array of Boolean variables X
set variable S

```
channel(home, X, S)
```

$$X_i = 1 \iff i \in S \quad 0 \leq i < |X|$$

Example

```
S = {1,2}
X = [1,1,0]
```

Constraints on FS variables
Channeling constraints

X an array of integer variables,
SA an array of set variables

```
channel(home, X, SA)
```

$$X_i = j \iff i \in SA_j \quad 0 \leq i, j < |X|$$

$$SA_i = s \iff \forall j \in s : X_j = i$$

Example

```
SA = [{1,2},{3}]
X = [1,1,2]
```

Constraints on FS variables
Channeling constraints

An array of integer variables $\vec{x}$
a set variable S:

```
rel(home, SOT_UNION, x, S)
```

constrains S to be the set $\{x_0, \ldots, x_{|x|-1}\}$

```
channelSorted(home, x, S);
```

constrains S to be the set $\{x_0, \ldots, x_{|x|-1}\}$, and the integer variables in $\vec{x}$ to be sorted in increasing order ($x_i < x_{i+1}$ for $0 \leq i < |x|$)

Example

```
rel(home, SOT_UNION, [3,6,2,1], {1,2,3,6})
channelSorted(home, [1,2,3,6], {1,2,3,6})
```

Constraints on FS variables
Channeling constraints

SA_1 and SA_2 two arrays of set variables

```
channel(home, SA1, SA2)
```

$$SA_1[i] = s \iff \forall j \in s : i \in SA_2[j]$$

$$\begin{aligned} SA_1[i] &= \{j \mid SA_2[j] \text{ contains } i\} \\ SA_2[j] &= \{i \mid SA_1[i] \text{ contains } j\} \end{aligned}$$

Example

```
SA1 = [{1,2},{3},{1,2}]
SA2 = [{1,3},{1,3},{2}]
```

Constraints on FS variables
Convexity

set variable S:

```
convex(home, S)
```

The convex hull of a set S is the smallest convex set containing S

```
convex(home, S1, S2)
```

enforces that the set variable $S2$ is the convex hull of the set variable $S1$.

Example

```
S={{1,2,5,6,7},{2,3,4},{3,5}}  convex(S)={2,3,4}
convex({1,2,5,6,7},{1,2,3,4,5,6,7})
```

Constraints on FS variables

Sequence constraints

enforce an order among an array of set variables x

```
sequence(home,x)
```

sets x being pairwise disjoint, and furthermore $\max(x_i) < \min(x_{i+1})$ for all $0 \leq i < |x| - 1$

```
sequence(home, x, y)
```

additionally constrains the set variable y to be the union of the x.

Constraints on FS variables
Value precedence constraints

enforce that a value precedes another value in an array of set variables.
x is an array of set variables and both s and t are integers,

```
precede(home, x, s, t)
```

if there exists j ($0 \leq j < |x|$) such that $s \notin x_j$ and $t \in x_j$, then there must exist i with $i < j$ such that $s \in x_i$ and $t \notin x_i$

Constraint Programming
Model with Set Variables

```
p = 9 # number of players
g = 3 # number of groups
w = 4 # number of weeks

s = p/g # size of groups

#=== Variables ===============
groups = setvars(g*w, intset(), 0, p-1, s, s)
schedule = Matrix(g, w, groups)
allPlayers = setvar(0, p-1, 0, p-1)

#=== Constraints ============
# In each week, groups must be disjoint and contain all players
for i in range(g):
   z1 = setvars(g, intset(), 0, p-1, 0, p)
   rel(SOT_DUNION, schedule[i].row(i), z1[i])
   rel(z1[i], SRT_EQ, allPlayers)

# at most one player overlaps between groups
for i,j in itertools.combinations(range(g*w), 2):
   z2 = setvar(intset(), 0, p-1, 0, p))
   rel(groups[i], SOT_INTER, groups[j], SRT_EQ, z2)
   cardinality(z2, 0, 1)

dom(groups[0],SRT_EQ,intset(0,2)) # {0,1,2} in groups[0] to break symmetry
branch(groups, SET_VAR_MIN_MIN, SET_VAL_MIN_INC);
```

Set Domain representation

- A finite integer set V can be represented by its characteristic function χ_V:

$$\chi_V : \mathbb{Z} \mapsto \{0, 1\} \text{ where } \chi_v(i) = 1 \text{ iff } i \in V$$

  hence we can use a set of Boolean variables v_i to represent the set V, which correspond to the propositions $v_i \iff i \in V$

  Set bounds propagation is equivalent to performing domain propagation in a naive way on this Boolean representation

- Sets of sets: disjunction of characteristic functions

$$\chi_{\mathcal{V}}(i) \iff \bigvee_{V \in \mathcal{V}} \chi_V(i)$$

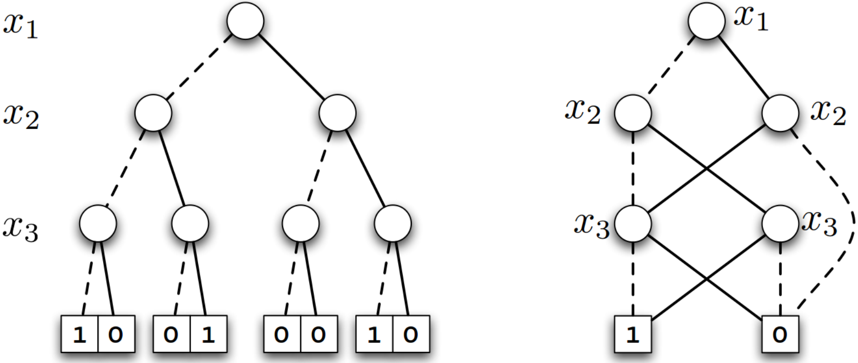
- Consider the domain $\{\{\}, \{1, 2\}, \{2, 3\}\}$
- Introduce propositional variables x_1, x_2, x_3
- Represent single variable domain as

$$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3))$$

- Represent all variable domains as conjunction
- Efficient datastructure: ROBDDs

ROBDD

A Reduced Ordered Binary Decision Diagram (ROBDD) is a compact data structure:
a canonical function representation up to reordering, which permits an efficient implementation of many Boolean function operations.



Implementation in Gecode

- *Set variables in Gecode do not use Reduced Ordered Binary Decision Diagrams (ROBDDs).*
- *A prototype alternative implementation using ROBDDs proved to be a lot slower in many cases (and quite painful to maintain because of additional dependencies).*
- *The current implementation uses range lists (i.e. linked lists of contiguous, sorted, non-overlapping ranges) to store a lower and an upper bound, together with a lower and upper bound on the cardinality.*

Guido Tack

Outline

Graph Variables

Definition

A graph variable is simply two set variables V and E, with an inherent constraint $E \subseteq V \times V$.

Hence, the domain $D(G) = [lb(G), ub(G)]$ of a graph variable G consists of:

- mandatory vertices and edges $lb(G)$ (the lower bound graph) and
- possible vertices and edges $ub(G) \setminus lb(G)$ (the upper bound graph).

The value assigned to the variable G must be a subgraph of $ub(G)$ and a super graph of the $lb(G)$.

Bound consistency on Graph Variables

Graph variables are convinient for possiblity of efficient filtering algorithms

Example:

`Subgraph(G,S)`

specifies that S is a subgraph of G. Computing bound consistency for the subgraph constraint means the following:

1. If $lb(S)$ is not a subgraph of $ub(G)$, the constraint has no solution (consistency check).
2. For each $e \in ub(G) \cap lb(S)$, include e in $lb(G)$.
3. For each $e \in ub(S) \setminus ub(G)$, remove e from $ub(S)$.

Constraints on Graph Variables

- Tree constraint: enforces the partitioning of a digraph into a set of vertex-disjoint anti-arborescences. (see, [Beldiceanu2005])
- Weghted Spanning Tree constraint: given a weighted undirected graph $G = (V, E)$ and a weight K, the constraint enforces that T is a spanning tree of cost at most K (see, [Regin2008,2010] and its application to the TSP [Rousseau2010]).
- Shorter Path constraint: given a weighted directed graph $G = (N, A)$ and a weight K, the constraint specifies that P is a subset of G, corresponding to a path of cost at most K. (see, [Sellmann2003, Gellermann2005])
- (Weighted) Clique Constraint, (see, [Regin2003]).

Outline

1. Set Variables
2. Graph Variables
3. Float Variables

Float Variables

- Floating point values represented as a closed interval of two floating point numbers (short, float number):
  closed interval $[a..b]$ to represent all real numbers n such that $a \leq n \leq b$.
- correct computations: no possible real number is ever excluded due to rounding $\rightsquigarrow$ Interval arithmetic
- The float number type `FloatNum` defined as double
- `FloatVar x; x.min(); x.max(); x.tight()` ($a = b$ assigned)
- predefined values `pi_half(), pi(), pi_twice()`
- `x<y` $\rightsquigarrow$ `x.max()<y.min()`

function	meaning	default
`max(x,y)`	maximum max(x, y)	✓
`min(x,y)`	minimum max(x, y)	✓
`abs(x)`	absolute value $\lvert x \rvert$	✓
`sqrt(x)`	square root $\sqrt{x}$	✓
`sqr(x)`	square x^2	✓
`pow(x,n)`	n-th power x^n	✓
`nroot(x,n)`	n-th root $\sqrt[n]{x}$	✓
`fmod(x,y)`	remainder of x/y	
`exp(x)`	exponential exp(x)	
`log(x)`	natural logarithm log(x)	
`sin(x)`	sine sin(x)	
`cos(x)`	cosine cos(x)	
`tan(x)`	tangent tan(x)	
`asin(x)`	arcsine arcsin(x)	
`acos(x)`	arccosine arccos(x)	
`atan(x)`	arctangent arctan(x)	
`sinh(x)`	hyperbolic sine sinh(x)	
`cosh(x)`	hyperbolic cosine cosh(x)	
`tanh(x)`	hyperbolic tangent tanh(x)	
`asinh(x)`	hyperbolic arcsine arcsinh(x)	
`acosh(x)`	hyperbolic arccosine arccosh(x)	

Variable Creation

```
FloatVar x(home, -1.0, 1.0); // creation
FloatVar y(x); // call to copy constructor, refer to variable x
FloatVar z; // default constructor, no variable implemented
z=y; // copy, z refer to x
cout<<x;
```

The variables x, y, and z all refer to the same float variable implementation.

Constraints

```
dom(home, x, -2.0, 12.0);
dom(home, x, d);

rel(home, x, FRT_LE, y);
rel(home, x, FRT_LQ, 4.0);

rel(home, x, FRT_LQ, y);
rel(home, x, FRT_GR, 7.0);

min(home, x, y);

linear(home, a, x, FRT_EQ, c);
linear(home, x, FRT_GR, c);

channel(home, x, y);
```

Interval Arithmetics

Whereas classical arithmetic defines operations on individual numbers,
interval arithmetic defines a set of operations on intervals:
For intervals on integers:

$$T \cdot S = \{x \mid \text{ there is some } y \text{ in } T, \text{ and some } z \text{ in } S, \text{ such that } x = y \cdot z\}.$$

For intervals on real numbers, the arithmetic is an extension of real arithmetic.
Let two intervals $[a, b]$ and $[c, d]$ be subsets of the real line $(-\infty, +\infty)$:

Definition

If $*$ is one of the symbols $+, -, \cdot, /$ for the arithmetic operations on intervals, then

$$[a, b] * [c, d] = \{x * y \mid a \leq x \leq b, c \leq y \leq d\}$$

except that $[a, b]/[c, d]$ remains undefined if $0 \in [c, d]$.

From the definition:

- $[a, b] + [c, d] = [a + c, b + d]$,
- $[a, b] - [c, d] = [a - d, b - c]$,
- $[a, b] \times [c, d] = [\min(a \times c, a \times d, b \times c, b \times d), \max(a \times c, a \times d, b \times c, b \times d)]$,
- $[a, b]/[c, d] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$ when 0 is not in $[c, d]$.

The addition and multiplication operations are commutative, associative and sub-distributive: the set $X(Y + Z)$ is a subset of $XY + XZ$.

See [Apt, 2003, sc 6.6]

References

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