DM545

Linear and Integer Programming

Lecture 6
Sensitivity Analysis and Farkas Lemma

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Outline

1. Geometric Interpretation

2. Sensitivity Analysis

3. Farkas Lemma

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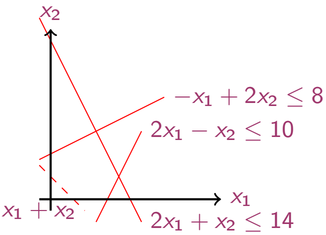
Geometric Interpretation

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 2x_1 + x_2 \leq 14 \\ & -x_1 + 2x_2 \leq 8 \\ & 2x_1 - x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

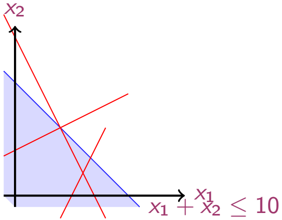


Opt $x^* = (4, 6)$, $z^* = 10$. To prove this we need to prove that $y^* = (3/5, 1/5, 0)$ is a feasible solution of D:

$$\begin{aligned} \min \ 14y_1 + 8y_2 + 10y_3 &= w \\ 2y_1 - y_2 + 2y_3 &\geq 1 \\ y_1 + 2y_2 - y_3 &\geq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

and that $w^* = 10$

$$\begin{array}{rl} \frac{3}{5}\cdot & 2x_1 + x_2 \le 14 \\ \frac{1}{5}\cdot & -x_1 + 2x_2 \le 8 \\ \hline & x_1 + x_2 \le 10 \end{array}$$



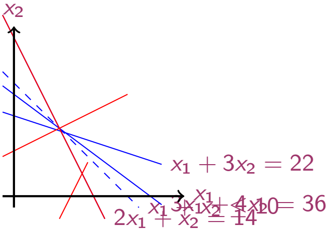
the feasibility region of P is a subset of the half plane $x_1 + x_2 \le 10$

$(2v - w)x_1 + (v + 2w)x_2 \le 14v + 8w$ set of half planes that contain the feasibility region of P and pass through $[4, 6]$

$$2v - w \ge 1$$
$$v + 2w \ge 1$$

Example of boundary lines among those allowed:

$$v = 1, w = 0 \implies 2x_1 + x_2 = 14$$
$$v = 1, w = 1 \implies x_1 + 3x_2 = 22$$
$$v = 2, w = 1 \implies 3x_1 + 4x_2 = 36$$



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Sensitivity Analysis
aka Postoptimality Analysis

Instead of solving each modified problems from scratch, exploit results obtained from solving the original problem.

$$\max\{c^T x \mid Ax = b, l \leq x \leq u\} \tag{*}$$

(I) changes to coefficients of objective function:
$\max\{\tilde{c}^T x \mid Ax = b, l \leq x \leq u\}$ (primal)
x^* of (*) remains feasible hence we can restart the simplex from x^*

(II) changes to RHS terms: $\max\{c^T x \mid Ax = \tilde{b}, l \leq x \leq u\}$ (dual)
x^* optimal feasible solution of (*)
basic sol $\bar{x}$ of (II): $\bar{x}_N = x^*_N$, $A_B \bar{x}_B = \tilde{b} - A_N \bar{x}_N$
$\bar{x}$ is dual feasible and we can start the dual simplex from there. If $\tilde{b}$ differs from b only slightly it may be we are already optimal.

(III) introduce a new variable: (primal)

$$\max \sum_{j=1}^{6} c_j x_j \qquad\qquad \max \sum_{j=1}^{7} c_j x_j$$

$$\sum_{j=1}^{6} a_{ij} x_j = b_i,\ i = 1, \dots, 3 \qquad\qquad \sum_{j=1}^{7} a_{ij} x_j = b_i,\ i = 1, \dots, 3$$

$$l_j \leq x_j \leq u_j,\ j = 1, \dots, 6 \qquad\qquad l_j \leq x_j \leq u_j,\ j = 1, \dots, 7$$

$[x_1^*, \dots, x_6^*]$ feasible $\qquad\qquad$ $[x_1^*, \dots, x_6^*, 0]$ feasible

(IV) introduce a new constraint: (dual)

$$\sum_{j=1}^{6} a_{4j} x_j = b_4$$

$$\sum_{j=1}^{6} a_{5j} x_j = b_5$$

$$l_j \leq x_j \leq u_j \qquad j = 7, 8$$

$[x_1^*, \dots, x_6^*]$ optimal

$[x_1^*, \dots, x_6^*, x_7^*, x_8^*]$ feasible

$$x_7^* = b_4 - \sum_{j=1}^{6} a_{4j} x_j^*$$

$$x_8^* = b_5 - \sum_{j=1}^{6} a_{5j} x_j^*$$

Examples

(I) Variation of reduced costs:

$$\begin{aligned} \max\ 6x_1 + \ 8x_2 & \\ 5x_1 + 10x_2 &\leq 60 \\ 4x_1 + \ 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

The last tableau gives the possibility to estimate the effect of variations

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

For a variable in basis the perturbation goes unchanged in the red. costs. Eg:

$$\max (6+\delta)x_1 + 8x_2 \implies \bar{c}_1 = -\frac{2}{5}\cdot 5 - 1 \cdot 4 + 1(6+\delta) = \delta$$

then need to bring in canonical form and hence δ changes the obj value. For a variable not in basis, if it changes the sign of the reduced cost $\implies$ worth bringing in basis $\implies$the δ term propagates to other columns

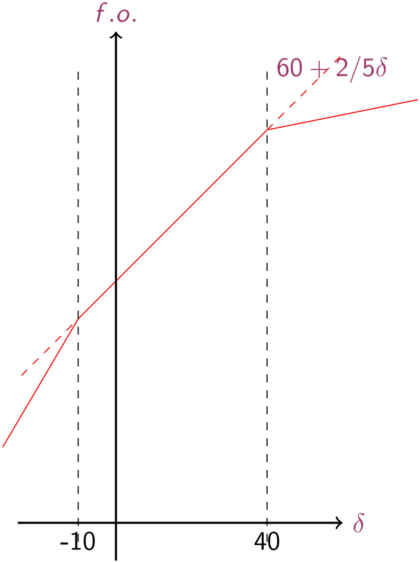
(II) Changes in RHS terms

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	10	1	0	0	$60+\delta$
x_4	4	4	0	1	0	$40+\epsilon$
	6	8	0	0	1	0

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	$1/5$	$-1/4$	0	$2+1/5\delta-1/4\epsilon$
x_1	1	0	$-1/5$	$1/2$	0	$8-1/5\delta+1/2\epsilon$
	0	0	$-2/5$	-1	1	$-64-2/5\delta-\epsilon$

(It would be more convenient to augment the second. But let's take $\epsilon = 0$.)
If $60+\delta$ $\Longrightarrow$all RHS terms change and we must check feasibility
Which are the multipliers for the first row?$k_1 = \frac{1}{5}, k_2 = -\frac{1}{4}, k_3 = 0$
I: $1/5(60+\delta) - 1/4 \cdot 40 + 0 \cdot 0 = 12 + \delta/5 - 10 = 2 + \delta/5$
II: $-1/5(60+\delta) + 1/2 \cdot 40 + 0 \cdot 0 = -60/5 + 20 - \delta/5 = 8 - 1/5\delta$
Risk that RHS becomes negative
Eg: if $\delta = -20$ $\Longrightarrow$tableau stays optimal but not feasible $\Longrightarrow$apply dual simplex

Graphical Representation



(III) Add a variable

$$
\begin{aligned}
\max\ 5x_0 + 6x_1 + \ 8x_2 & \\
6x_0 + 5x_1 + 10x_2 &\leq 60 \\
8x_0 + 4x_1 + \ 4x_2 &\leq 40 \\
x_0, x_1, x_2 &\geq \ 0
\end{aligned}
$$

Reduced cost of x_0? $c_j + \sum \pi_i a_{ij} = +1 \cdot 5 - \frac{2}{5} \cdot 6 + (-1)8 = -\frac{27}{5}$

To make worth entering in basis:

- increase its cost
- decrease the amount in constraint II: $-2/5 \cdot 6 - a_{20} + 5 > 0$

(IV) Add a constraint

$$\begin{aligned} \max\ & 6x_1 + 8x_2 \\ & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & 5x_1 + 6x_2 \leq 50 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Final tableau not in canonical form, need to iterate

	x_1	x_2	x_3	x_4	x_5	$-z$	b
x_2	0	1	$1/5$	$-1/4$		0	2
x_1	1	0	$-1/5$	$1/2$		0	8
	0	0	$5/5$	$6/4$	1	0	-2
	0	0	$-2/5$	-1	0	1	-64

(V) change in a technological coefficient:

	x_1	x_2	x_3	x_4	$-z$	b
x_3	5	$10+\delta$	1	0	0	60
x_4	4	4	0	1	0	40
	6	8	0	0	1	0

- first effect on its column
- then look at c
- finally look at b

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	$(10+\delta)1/5+4(-1/4)$	$1/5$	$-1/4$	0	2
x_1	1	$(10+\delta)(-1/5)+4(1/2)$	$-1/5$	$1/2$	0	8
	0	$-2/5\delta$	$-2/5$	-1	1	-64

The dominant application of LP is mixed integer linear programming. In this context it is extremely important being able to begin with a model instantiated in one form followed by a sequence of problem modifications (such as row and column additions and deletions and variable fixings) interspersed with resolves

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Strong Duality

Summary of Proof seen earlier in matrix notation:

Assuming that P and D have feasible solutions:
there exists an optimal basis B and an optimal solution $\mathbf{x}_B$
Dual solution corresponding to B, $\mathbf{y}_B = \mathbf{c}_B^T \mathbf{A}_B{}^{-1}$, aka multipliers for B
From the simplex:

$$\bar{\mathbf{c}} = \mathbf{c} + \pi \mathbf{A}$$

and at optimality $\mathbf{c}_B = \mathbf{0}$ for basic variables and $\mathbf{c}_{\bar{B}} \geq \mathbf{0}$ for non basic variables
Setting $\mathbf{y}_B = -\pi$ we obtain $\mathbf{y}_B \mathbf{A} \leq \mathbf{c}$ and hence $\mathbf{y}_B$ is feasible for the dual.
What is the value of this dual solution?

$$\mathbf{y}_B^T \mathbf{b} = \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}^T \mathbf{x}$$

We now look at Farkas Lemma with two objectives:

- giving another proof of strong duality
- understanding a certificate of infeasibility

Farkas Lemma

Lemma (Farkas)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then,

either	*I.*	$\exists x \in \mathbb{R}^n : Ax = b$ and $x \geq 0$
or	*II.*	$\exists y \in \mathbb{R}^m : y^T A \geq 0^T$ and $y^T b < 0$

Easy to see that both I and II cannot occur together:

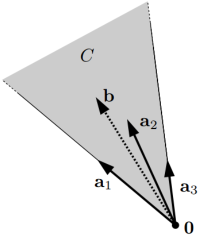
$$(0 \leq) \quad \underbrace{(y^T A)}_{\geq 0} \underbrace{x}_{\geq 0} y^T Ax = y^T b \quad (< 0)$$

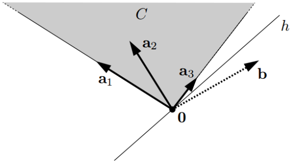
Geometric interpretation of Farkas L.

Linear combination of a_i with nonnegative terms generates a convex cone:

$$\{\lambda_1 a_1 + \ldots + \lambda_n a_n, \mid \lambda_1, \ldots, \lambda_n \geq 0\}$$

polyhedral cone: $C = \{x \mid Ax \leq 0\}$, intersection of many $ax \leq 0$
Convex hull of rays $p_i = \{\lambda_i a_i, \lambda_i \geq 0\}$





Either point b lies in convex cone C

or $\exists$ hyperplane h passing through point 0 $h = \{x \in \mathbb{R}^m : y^T x = 0\}$ for $y \in \mathbb{R}^m$ such that all vectors $a_1, \ldots, a_n$ (and thus C) lie on one side and b lies (strictly) on the other side (ie, $y^T a_i \geq 0, \forall i = 1 \ldots n$ and $y^T b < 0$).

Variants of Farkas Lemma

Corollary

(i) $\mathbf{Ax} = \mathbf{b}$ *has sol* $\mathbf{x} \geq \mathbf{0} \iff \forall \mathbf{y} \in \mathbb{R}^m$ *with* $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

(ii) $\mathbf{Ax} \leq \mathbf{b}$ *has sol* $\mathbf{x} \geq \mathbf{0} \iff \forall \mathbf{y} \geq \mathbf{0}$ *with* $\mathbf{y}^T\mathbf{A} \geq \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

(iii) $\mathbf{Ax} \leq \mathbf{0}$ *has sol* $\mathbf{x} \in \mathbb{R}^n \iff \forall \mathbf{y} \geq \mathbf{0}$ *with* $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \geq \mathbf{0}$

i) $\implies$ ii):

$\bar{\mathbf{A}} = [\mathbf{A} \mid I_m]$

$\mathbf{Ax} \leq \mathbf{b}$ has sol $\mathbf{x} \geq \mathbf{0} \iff \bar{\mathbf{A}}\mathbf{x} = \mathbf{b}$ has sol $\mathbf{x} \geq \mathbf{0}$

By (i):

$\forall \mathbf{y} \in \mathbb{R}^m$
$\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$, $\mathbf{y}^T\bar{\mathbf{A}} \geq \mathbf{0}$

$\mathbf{y}^T\mathbf{A} \geq \mathbf{0}$
$\mathbf{y} \geq \mathbf{0}$

relation with Fourier & Moutzkin method

	The system $A\mathbf{x} \leq \mathbf{b}$	The system $A\mathbf{x} = \mathbf{b}$
has a solution $\mathbf{x} \geq \mathbf{0}$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A \geq \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A \geq \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$
has a solution $\mathbf{x} \in \mathbb{R}^n$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A = \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} = 0$

Strong Duality by Farkas Lemma

Assume P has opt $\mathbf{x}^*$ and find D has opt as well.
Opt value for P:

$$\gamma = \mathbf{c}^T \mathbf{x}^*$$

We know by assumption:

$\begin{matrix} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{c}^T\mathbf{x} \geq \gamma \end{matrix}$ has sol $\mathbf{x} \geq \mathbf{0}$

and $\forall \epsilon > 0$
$\begin{matrix} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{c}^T\mathbf{x} \geq \gamma + \epsilon \end{matrix}$ has no sol $x \geq \mathbf{0}$

Let's define:

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ -\mathbf{c}^T \end{bmatrix} \qquad \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -\gamma - \epsilon \end{bmatrix}$$

and consider $\hat{\mathbf{A}}\mathbf{x} \leq \hat{\mathbf{b}}_0$ and $\hat{\mathbf{A}}\mathbf{x} \leq \hat{\mathbf{b}}_\epsilon$

we apply variant (ii) of Farkas' Lemma:

For $\epsilon \geq 0$, $\hat{\mathbf{A}}\mathbf{x} \leq \hat{\mathbf{b}}_\epsilon$ has no sol $\mathbf{x} \geq \mathbf{0}$ is equivalent to:
there exists $\hat{\mathbf{y}} = (\mathbf{u}, z) \in \mathbb{R}^{m+1}$,

$$\begin{aligned}\hat{\mathbf{y}} &\geq \mathbf{0}\\ \hat{\mathbf{y}}^T\hat{\mathbf{A}} &\geq \mathbf{0}\\ \hat{\mathbf{y}}^T\mathbf{b}_\epsilon &< 0\end{aligned}$$

Then

$$\begin{aligned}\mathbf{A}^T\mathbf{u} &\geq \mathbf{0}\\ \mathbf{b}^T\mathbf{u} &< z(\gamma + \epsilon)\end{aligned}$$

For $\epsilon = 0$, $\hat{\mathbf{A}}\mathbf{x} \leq \hat{\mathbf{b}}_0$ has sol $\mathbf{x} \geq \mathbf{0}$ is equivalent to:
there exists $\hat{\mathbf{y}} = (\mathbf{u}, z) \in \mathbb{R}^{m+1}$,

$$\begin{aligned}\hat{\mathbf{y}} &\geq \mathbf{0}\\ \hat{\mathbf{y}}^T\hat{\mathbf{A}} &\geq \mathbf{0}\\ \hat{\mathbf{y}}^T\mathbf{b}_0 &\geq 0\end{aligned}$$

Then

$$\begin{aligned}\mathbf{A}^T\mathbf{u} &\geq \mathbf{0}\\ \mathbf{b}^T\mathbf{u} &\geq z\gamma\end{aligned}$$

Hence, $z > 0$ or $z = 0$ would contradict the separation of cases.

We can set $\mathbf{v} = 1/z\mathbf{u} \geq 0$

$$\begin{aligned}\mathbf{A}^T\mathbf{v} &\geq \mathbf{c}\\ \mathbf{b}^T\mathbf{v} &< \gamma + \epsilon\end{aligned}$$

$\mathbf{v}$ is feasible sol of D with objective value $< \gamma + \epsilon$

By weak duality γ is upper bound. Since D bounded and feasible then there exists $\mathbf{y}^*$:

$$\gamma \leq \mathbf{b}^T\mathbf{y}^* < \gamma + \epsilon \qquad \forall \epsilon > 0$$

which implies $\mathbf{b}^T\mathbf{y}^* = \gamma$

Certificate of Infeasibility

Farkas Lemma provides a way to certificate infeasibility.
Given a certificate y^* it is easy to check the conditions (by linear algebra):

$$A^T y^* \geq 0$$
$$by^* < 0$$

Why y^* would be a certificate of infeasibility?
Proof: (by contradiction)
Assume, $A^T y^* \geq 0$ and $by^* < 0$.
Moreover assume $\exists x^*$: $Ax^* = b$, $x^* \geq 0$,then:

$$(\geq 0) \qquad (y^*)^T A x^* = (y^*)^T b \qquad (< 0)$$

Contradiction

General form:

$$\begin{aligned} \max\ & c^T x \\ A_1 x &= b_1 \\ A_2 x &\leq b_2 \\ A_3 x &\geq b_3 \\ x &\geq 0 \end{aligned}$$

infeasible $\Leftrightarrow \exists y^*$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 + b_3^T y_3 &> 0 \\ A_1^T y_1 + A_2^T y_2 + A_3^T y_3 &\leq 0 \\ y_2 &\leq 0 \\ y_3 &\geq 0 \end{aligned}$$

Example:

$$\begin{aligned} \max\ & c^T x \\ x_1 &\leq 1 \\ x_1 &\geq 2 \end{aligned}$$

$$\begin{aligned} b_1^T y_1 + b_2^T y_2 &> 0 \\ A_1^T y_1 + A_2^T y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} y_1 + 2y_2 &> 0 \\ y_1 + y_2 &\leq 0 \\ y_1 &\leq 0 \\ y_2 &\geq 0 \end{aligned}$$

$y_1 = -1, y_2 = 1$ is a valid certificate.

- Observe that it is not unique!
- It can be reported in place of the dual solution because same dimension.
- To repair infeasibility we should change the primal at least so much as that the certificate of infeasibility is no longer valid.
- Only constraints with $y_i \neq 0$ in the certificate of infeasibility cause infeasibility

Duality: Summary

- Derivation:
  1. bounding
  2. multipliers
  3. recipe
  4. Lagrangian (to do)
- Theory:
  - Symmetry
  - Weak duality theorem
  - Strong duality theorem
  - Complementary slackness theorem
  - Farkas Lemma:
    Strong duality + Infeasibility certificate
- Dual Simplex
- Economic interpretation
- Geometric Interpretation
- Sensitivity analysis

Resume

Advantages of considering the dual formulation:

- proving optimality (although the simplex tableau can already do that)
- gives a way to check the correctness of results easily
- alternative solution method (ie, primal simplex on dual)
- sensitivity analysis
- solving P or D we solve the other for free
- certificate of infeasibility