DM865 – Heuristics & Approximation Algorithms

(Marco) (Lene)

Prerequisites

Required: Programming
Alg. & Datastructures

Recommended: Complexity & Computability
Linear & Integer Prg.

3 lectures per week (1-2 during project work)

Project in two parts
- 2-3 weeks per part
- 2 students per group
- Vehicle routing

Oral exam, 7 mark scale
- 10 min about project
- 10 min about other topics from the course

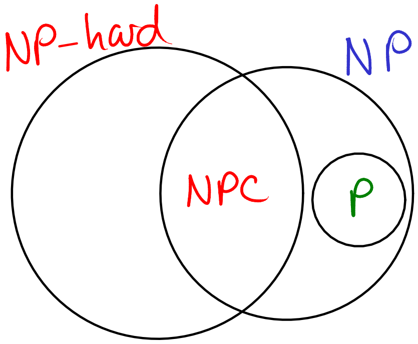
Combinatorial problems:

Set Cover (today)
Traveling Salesman (TSP)
SAT
Knapsack
Scheduling
Bin packing

$\left.\right\}$ decision version $\in$ NPC

Polynomial algorithm:

algo. with running time $O(n^c)$, for some constant c.



P: The set of decision problems that allow for a poly. algo.

NP: A problem belongs to NP, if solutions can be verified in poly. time.

If any NP-hard problem has a poly. algo., then all problems in NPC have poly. algo.s.

(1) Optimal solutions
(2) in poly. time
(3) for all instances
- Choose two! ((2) & (3))

An approximation algorithm comes with a performance guarantee:

Def 1.1: α-approximation algorithm

An α-approximation algorithm for an optimization problem P is a poly. time algo. ALG s.t. for any instance I of P,

- $\frac{ALG(I)}{OPT(I)} \leq \alpha$, if P is a minimization problem
- $\frac{ALG(I)}{OPT(I)} \geq \alpha$, if P is a maximization problem

Thus, for max. problems, $0 \leq \alpha \leq 1$, and, for min. problems, $\alpha \geq 1$.

The approximation factor / approximation ratio is
- the smallest possible α (for min. problems)
- the largest possible α (for max. problems)

Techniques: (with Set Cover as an example)

- Solve LP and round solution (Sec. 1.3 + 1.7)
- Primal-dual alg.: combinatorial alg. based on LP formulation (Sec. 1.4 + 1.5)
- Greedy alg. (Sec. 1.6)

Section 1.2: Set Cover as an LP

Set Cover

Input:

$E = \{e_1, e_2, \dots, e_n\}$

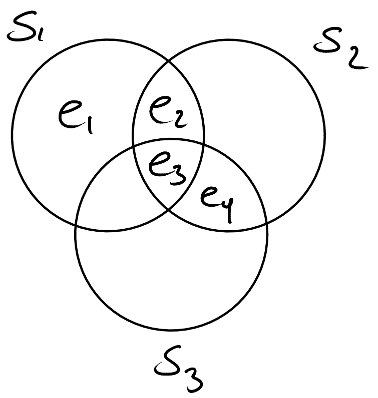
$\mathcal{S} = \{S_1, S_2, \dots, S_m\}$, where

$S_j \subseteq E$ has weight w_j.

Objective: Find a cheapest possible subset of $\mathcal{S}$ covering all elements

OPT: value (total weight) of optimum solution

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

$\{S_1, S_2\}$ is a sol. of total weight 3.

This is optimal, so OPT $= 3$ for this instance of Set Cover.

To cover e_1, we need S_1

—— " —— e_2 —— " —— S_1 or S_2

—— " —— e_3 —— " —— S_1 or S_3

—— " —— e_4 —— " —— S_2 or S_3

IP-formulation:

$$\begin{array}{ll} \min & x_1 w_1 + x_2 w_2 + x_3 w_3 \\ \text{s.t.} & x_1 \geq 1 \\ & x_1 + x_2 \geq 1 \\ & x_1 + x_2 + x_3 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1, x_2, x_3 \in \{0,1\} \end{array}$$

More generally:

IP for Set Cover

$$\begin{aligned} \min \quad & \sum_{j=1}^{m} x_j w_j \\ \text{s.t.} \quad & \sum_{j : e_i \in S_j} x_j \geq 1 , && i = 1, 2, \ldots, n \\ & x_j \in \{0,1\} , && j = 1, 2, \ldots, m \end{aligned}$$

Z^*_{IP} : optimum solution value, i.e., $Z^*_{IP} = OPT$

LP- relaxation

$$\begin{aligned} \min \quad & \sum_{j=1}^{m} x_j w_j \\ \text{s.t.} \quad & \sum_{j : e_i \in S_j} x_j \geq 1 , && i = 1, 2, \ldots, n \\ & 0 \leq x_j \underset{\text{redundant}}{\leq 1} , && j = 1, 2, \ldots, m \end{aligned}$$

Z^*_{LP} : Optimum solution value

Note that $\boxed{Z^*_{LP} \leq Z^*_{IP} = OPT}$

Section 1.3: A deterministic rounding algo.

The frequency of an element e is the #sets containing e:

$$f_e = |\{S \in \mathcal{S} \mid e \in S\}|$$

The frequency of an instance of Set Cover:

$$f = \max_{e \in E} \{f_e\}$$

Alg. 1 for Set Cover: LP-rounding

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Solve LP
$I \leftarrow \{ j \mid x_j \geq \frac{1}{f} \}$
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We prove that Alg 1.1 produces a set cover (Lemma 1.5) of total weight $\leq f \cdot OPT$ (Thm 1.6)

Lemma 1.5

$\{ S_j \mid j \in I \}$ is a set cover

Proof:

For each $e_i \in E$, $\sum_{j: e_i \in S_j} x_j \geq 1$.

Since $\sum_{j: e_i \in S_j} x_j$ has at most f terms, at least one of the terms is at least $\frac{1}{f}$.

Thus, there is a set S_j s.t.
$e_i \in S_j$ and $x_j \geq \frac{1}{f}$.
This j is included in I ☐

Thm 1.6

Alg. 1 is an f-approx. algo. for Set Cover.

Proof:

Correct by Lemma 1.5
Poly, since LP-solving is poly.
Approx. factor f:
Each x_j is rounded up to 1, only if it is already at least $\frac{1}{f}$.
Thus, each x_j is multiplied by at most f, i.e.,

$$\sum_{j \in I} w_j \leq \sum_{j=1}^{m} f \cdot x_j \cdot w_j = f \cdot Z^*_{LP} \leq f \cdot OPT$$

☐

The Vertex Cover problem is a special case of Set Cover:

Vertex Cover

Input:

$G = (V, E)$

Objective:

Find a min. card. vertex set $C \subseteq V$ s.t. each edge $e \in E$ has at least one endpoint in C.

With $\mathcal{S} = V$ and $E = E$, Alg. 1 is a 2-approx. alg. for Vertex Cover.

Exercise for tomorrow:
Write down LP for Vertex Cover.

Section 1.4: The dual LP

What is a dual?

Ex: P

$$\boxed{\begin{array}{ll} \min & 7x_1 + x_2 + 5x_3 \\ \text{s.t.} & x_1 - x_2 + 3x_3 \geqslant 10 \\ & 5x_1 + 2x_2 - x_3 \geqslant 6 \\ & x_1, x_2, x_3 \geqslant 0 \end{array}}$$

Primal

$$7x_1 + x_2 + 5x_3 \geqslant x_1 - x_2 + 3x_3 \geqslant 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geqslant x_1 - x_2 + 3x_3 + 5x_1 + 2x_2 - x_3 \\ &\geqslant 10 + 6 = 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geqslant 2(x_1 - x_2 + 3x_3) + 5x_1 + 2x_2 - x_3 \\ &\geqslant 2 \cdot 10 + 6 = 26 \end{aligned}$$

To find a largest possible lower bound on $7x_1 + x_2 + 5x_3$, we should determine y_1 and y_2 maximizing $10y_1 + 6y_2$, under the constraints that

$$(*) \left\{ \begin{aligned} 7x_1 + x_2 + 5x_3 &\geqslant y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{\geqslant 10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{\geqslant 6} \\ &= (y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3 \end{aligned} \right.$$

and $\underbrace{y_1, y_2, y_3 \geqslant 0}_{\text{otherwise „}\geqslant\text{" becomes „}\leqslant\text{"}}$

Thus, we arrive at the following problem:

D

$$\begin{aligned} \max \quad & 10y_1 + 6y_2 \\ \text{s.t.} \quad & y_1 + 5y_2 \leq 7 \\ & -y_1 + 2y_2 \leq 1 \\ & 3y_1 - y_2 \leq 5 \\ & y_1, y_2 \geqslant 0 \end{aligned}$$

Dual

In general:

Primal:

$$\begin{aligned} \min \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geqslant b_i \;, \quad i = 1,2,\dots,m \\ & x_j \geqslant 0 \;, \quad j = 1,2,\dots,n \end{aligned}$$

Dual:

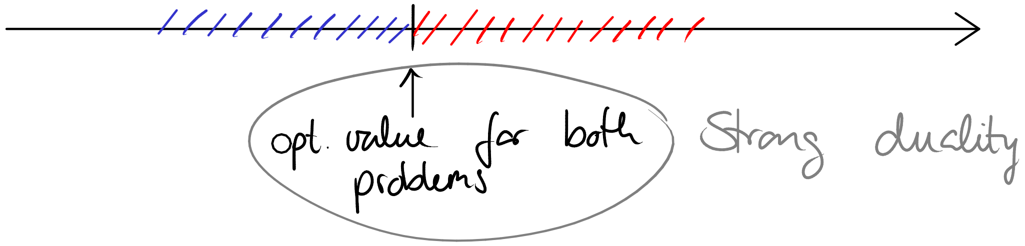
$$\begin{aligned} \max \quad & b_1 y_1 + b_2 y_2 + \dots + b_m y_m \\ \text{s.t.} \quad & a_{1j}y_1 + a_{2j}y_2 + \dots + a_{mj}y_m \leq c_1 \;, \quad j = 1,2,\dots,n \\ & y_i \geqslant 0, \quad i = 1,2,\dots,m \end{aligned}$$

Returning to the example above:

The constraints of D ensure that the value of any sol. to D is a lower bound on the value of any sol. to P, i.e., for any pair x, y of sol. to P and D resp.,

$$10y_1 + 6y_2 \leq 7x_1 + x_2 + 5x_3$$

Weak duality



Consider a pair $\vec{x}, \vec{y}$ of sol to P and D, resp.

If all constraints are tight, i.e.,

$$x_1 - x_2 + 3x_3 = 10 \quad \text{and} \quad 5x_1 + 2x_2 - x_3 = 6,$$

$$y_1 + 5y_2 = 7, \quad -y_1 + 2y_2 = 1, \quad \text{and} \quad 3y_1 - y_2 = 5,$$

then

$$7x_1 + x_2 + 5x_3 = \overbrace{(y_1 + 5y_2)}^{=7} x_1 + \overbrace{(-y_1 + 2y_2)}^{=1} x_2 + \overbrace{(3y_1 - y_2)}^{=5} x_3$$

Hence, by (*),

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &= y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{=10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{=6} \\ &= 10y_1 + 6y_2 \end{aligned}$$

Similarly, if, e.g.,

$$x_1 - x_2 + 3x_3 = 10 \quad \text{and} \quad y_2 = 0$$

$$x_1 = 0, \quad -y_1 + 2y_2 = 1, \quad \text{and} \quad 3y_1 - y_2 = 5,$$

then

$$\overbrace{7x_1}^{0} + x_2 + 5x_3 = \overbrace{(y_1 + 5y_2)}^{=0} x_1 + \overbrace{(-y_1 + 2y_2)}^{=1} x_2 + \overbrace{(3y_1 - y_2)}^{=5} x_3$$

Hence, by (*),

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &= y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{=10} + \overbrace{y_2 (5x_1 + 2x_2 - x_3)}^{=0} \\ &= 10y_1 + \underbrace{6y_2}_{=0} \end{aligned}$$

On the other hand:

If, e.g., $y_1 > 0$ and $x_1 - x_2 + 3x_3 > 10$, then

$$7x_1 + x_2 + 5x_3 \geqslant y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{>10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{\geqslant 6}$$
$$> 10y_1 + 6y_2$$

Similarly, if, e.g., $x_1 > 0$ and $y_1 + 5y_2 < 7$, then

$$10y_1 + 6y_2 \leqslant (x_1 - x_2 + 3x_3)y_1 + (5x_1 + 2x_2 - x_3)y_2$$
$$= \underbrace{(y_1 + 5y_2)}_{<7} x_1 + \underbrace{(-y_1 + 2y_2)}_{\leq 1} x_2 + \underbrace{(3y_1 - y_2)}_{\leq 5} x_3$$
$$< 7x_1 + x_2 + 5x_3$$

More generally:

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

$$\text{Complementary Slackness Conditions}\left\{\begin{array}{l} \left.\begin{array}{lcl} x_1 > 0 & \Rightarrow & y_1 + 5y_2 = 7 \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 = 1 \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 = 5 \end{array}\right\} \text{primal c.s.c.} \\ \\ \left.\begin{array}{lcl} y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 = 10 \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 = 6 \end{array}\right\} \text{dual c.s.c.} \end{array}\right.$$

By The Strong Duality Theorem (which we will not prove), there exist solutions fulfilling the c.s.c.

Moreover, if the c.s.c. are „close" to being satisfied, the values of the primal and dual sol. are „close":

$$\text{Relaxed Complementary Slackness Conditions}\left\{\begin{array}{lcl} x_1 > 0 & \Rightarrow & y_1 + 5y_2 \geq 7/b \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 \geq 1/b \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 \geq 5/b \\ & & \\ y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 \leq 10c \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 \leq 6c \end{array}\right.$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq bc(10y_1 + 6y_2)$$