Section 3.3 : The Bin Packing Problem

Last time we discussed simple approx. alg.s
Today we will develop an approximation scheme.

Approximation scheme $\{A_\varepsilon\}$:

1. Transform $I \to I''$:
   a. Remove all items smaller than $\varepsilon/2$. $(I \to I')$
      $\Rightarrow O(\frac{1}{\varepsilon})$ items fit in one bin
   b. Round up sizes of remaining items $(I' \to I'')$
      $\Rightarrow O(1)$ different item sizes
2. Do dyn. prg. on I''
   $\Rightarrow A_\varepsilon(I'') = OPT(I'')$
3. Add small items to the packing using First-Fit (or any other Anyfit alg.)

Adding small items to the packing

Lemma 3.10

$$A_\varepsilon(I) \leq \max\left\{A_\varepsilon(I''), \frac{2}{2-\varepsilon}\,\mathrm{size}(I)+1\right\}$$

Proof:

If no extra bin is needed for adding the small items, $A_\varepsilon(I) = A_\varepsilon(I'')$.

Otherwise, all bins, except possibly the last one, are filled to more than $1-\varepsilon/2$.

In this case,

$$A_\varepsilon(I) \leq \left\lceil \frac{\mathrm{size}(I)}{1-\varepsilon/2} \right\rceil \leq \frac{\mathrm{size}(I)}{1-\varepsilon/2} + 1$$

$$= \frac{2}{2-\varepsilon}\,\mathrm{size}(I) + 1$$

□

Rounding scheme

Last time we saw that a rounding scheme similar to the one we used for Knapsack would at best yield an approx. factor of 2.
Instead, we will use:

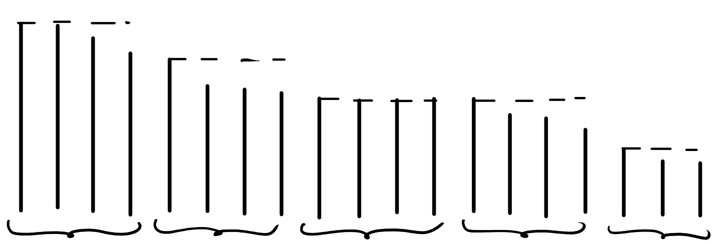
Linear grouping:

- Sort items in I' by decreasing sizes.
- Partition items in groups of k consecutive items.
  (k will be determined later)
- For each group, round up item sizes to largest size in the group.

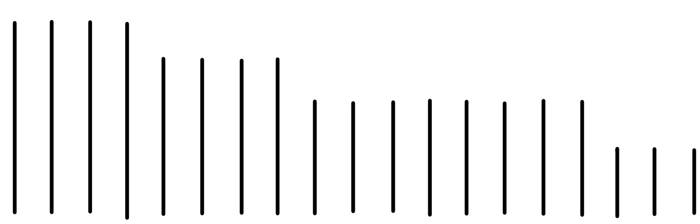
The result is called I''.

Ex: ($k=4$)

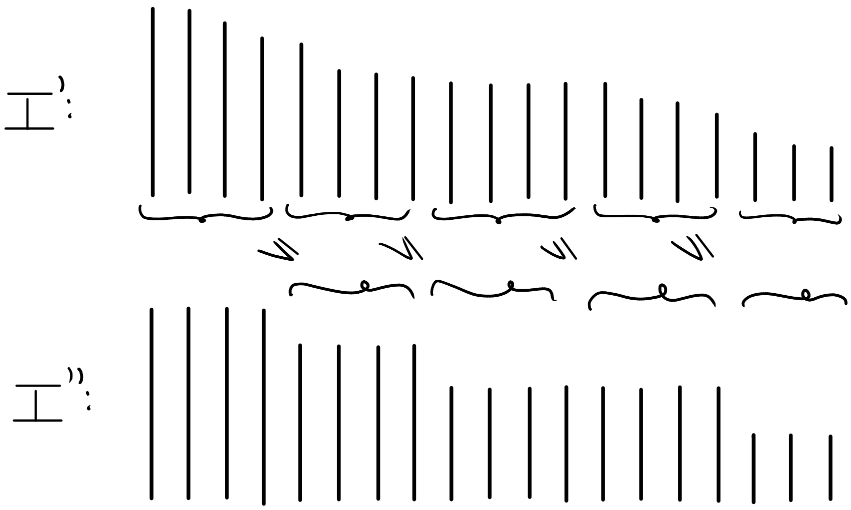
I':



I'':



Each item in the i'th group of I' is at least as large as any item in the (i+1)st group of I'':



Thus, for any packing of I', there is a packing of all but the first group of I'' using the same number of bins.
Since the first group of I'' can be packed in at most k bins, this proves:

$$\boxed{\text{Lemma 3.11: } OPT(I'') \le OPT(I') + k}$$

Approximation

$$A_\varepsilon(I) \le \max\left\{ A_\varepsilon(I''), \tfrac{2}{2-\varepsilon}\, size(I)+1 \right\}, \text{ by Lemma 3.10}$$

$$\le \max\left\{ OPT(I''), \tfrac{2}{2-\varepsilon}\, OPT(I)+1 \right\}, \text{ since}$$

$$A_\varepsilon(I'') = OPT(I'') \text{ and } OPT \geq size(I)$$

$$\le \max\left\{ OPT(I')+k, \tfrac{2}{2-\varepsilon}\, OPT(I)+1 \right\}, \text{ by Lemma 3.11}$$

$$\le \max\left\{ OPT(I)+k, \tfrac{2}{2-\varepsilon}\, OPT(I)+1 \right\}, \text{ since } I' \subseteq I$$

$$\frac{2}{2-\varepsilon} \le 1+\varepsilon \iff 2 \le (2-\varepsilon)(1+\varepsilon)$$

$$\iff 2 \le 2+\varepsilon-\varepsilon^2$$

$$\iff \varepsilon \le 1$$

Thus, we just need to choose an appropriate value of k to obtain $k \le \varepsilon \cdot OPT(I)$:

$$k = \lfloor \varepsilon \cdot Size(I) \rfloor$$

With this value of k

$$A_\varepsilon(I) \le (1+\varepsilon)\cdot OPT(I) + 1$$

asymptotic approximation scheme

There is no PTAS for Bin Packing:

Theorem 3.8

No approx alg. for Bin Packing has an approx. ratio better than $3/2$, unless $P = NP$.

Proof:

Reduction from Partition Problem (given a set S of integers, can S be partitioned into two sets S_1 and S_2 such that $\sum_{s \in S_1} s = \sum_{s \in S_2} s$?)

Let $B = \sum_{s \in S} s$.

Scale each integer by $\frac{2}{B}$, resulting in a set of numbers with sum 2.
Use these numbers as input for the bin packing problem.

Clearly, at least 2 bins are needed, and 2 bins are sufficient, if and only if the instance of the Partition problem is a yes-instance.

Thus, any Bin Packing alg. with an approx. ratio smaller than $3/2$ will use exactly 2 bins, if and only if the input to the Partition problem is a yes-instance. $\square$