Exercise 5.7:

Derandomize the rounding algorithm for Set Cover from Section 1.7.

Hint: Use the obj. fct. $W + \lambda Z$, for a proper choice of λ

$W = \sum_{j=1}^{m} X_j w_j$

random variable indicating whether the sol. is a set cover

Random variable indicating whether S_j is included in the sol.

Alg. RR_2 from Section 1.7:

$$Pr(Z=1) \le \frac{1}{n}$$

$$E[W] \le 2 \ln n \cdot Z^*_{LP} \le 2 \ln n \cdot OPT$$

Thus, letting $\lambda = n \ln n \cdot Z^*_{LP}$ and $f = W + \lambda Z$,

$$E[f] \le 3 \ln n \cdot Z^*_{LP}$$

$$Pr(X_j = 0) = (1 - x_j)^{2 \ln n}.$$

$\Downarrow$

$$Pr(X_j = 1) = 1 - (1 - x_j)^{2 \ln n} \qquad (1)$$

$$Pr(e_i \text{ } \underline{\text{not}} \text{ covered}) = \prod_{j : e_i \in S_j} (1 - x_j)^{2 \ln n}$$

$\Downarrow$

$$Pr(Z = 0) = \prod_{1 \le i \le n} \left(1 - \prod_{j : e_i \in S_j} (1 - x_j)^{2 \ln n} \right)$$

$\Downarrow$

$$Pr(Z=1) = 1 - \text{———————— '' ————————} \qquad (2)$$

$$E[f] = \sum_{1 \le j \le m} \Pr(X_j = 1) \cdot w_j + \lambda \cdot \Pr(Z=1)$$

Thus, $E[f]$ can be calculated using (1) and (2).

Let $f_{X_1 \cdots X_\ell} = \sum_{j=1}^{\ell} X_j w_j + f_{\ell+1}$, where $f_{\ell+1}$ is the function $f = W + \lambda Z$ for the problem where

- $S_1, \dots, S_\ell$ have been removed
- the items covered by the selected sets among $S_1, \dots, S_\ell$ have been removed.

DeRR$_2$

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Solve LP
For i ← 1 to m
    If E[f_{X_1...X_{i-1} 0}] ≤ E[f_{X_1...X_{i-1} 1}]
        X_i ← 0
    Else
        X_i ← 1
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Since $E[f] \leq 3 \cdot \ln n \cdot z^*_{LP}$, $DeRR_2$ returns a a sol. with $f \leq 3 \cdot \ln n \cdot z^*_{LP}$.

For $n > 3$ such a sol. is a valid set cover, since any nonvalid sol. has

$$f = W + \lambda Z \geq \lambda Z = \lambda = n \ln n \cdot z^*_{LP}$$

Furthermore, it has

$$W \leq f \leq 3 \cdot \ln n \cdot z^*_{LP} \leq 3 \cdot \ln n \cdot OPT$$

Hence, it is a $3 \cdot \ln n$ - approx. alg.