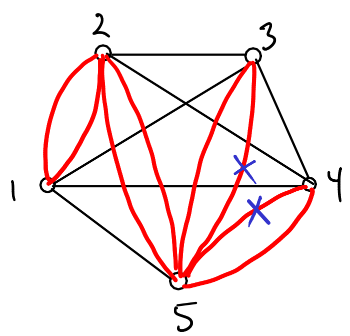
Double Tree algorithm

Noting that NA adds the edges of a MST one by one, we could also make a MST T and traverse T, making shortcuts whenever we would otherwise visit a node for the second time:

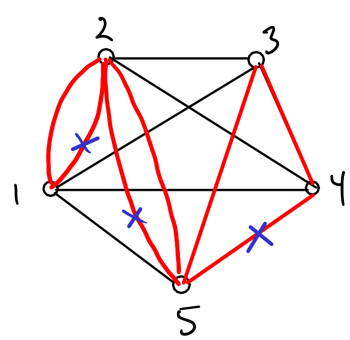


By the triangle inequality, this distance is no longer

$\langle 1, 2, 5, 3, \cancel{5}, 4, 5, 2, 1 \rangle$

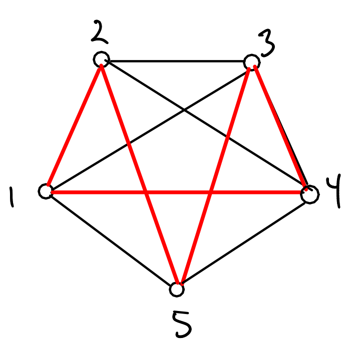
than this total distance

↓ shortcut 5



$\langle 1, 2, 5, 3, 4, \cancel{5}, \cancel{2}, 1 \rangle$

↓ shortcut 5 and 2 (using Δ-ineq. twice)



$\langle 1, 2, 5, 3, 4, 1 \rangle$

An Euler tour is a traversal of a graph that traverses each edge exactly once.

A graph that has an Euler tour is called eulerian.

A graph is eulerian if and only if all vertices have even degree.

Constructive proof of „if" in exercises for Wednesday.

Double Tree Algorithm (DT)

$T \leftarrow$ MST

$DT \leftarrow$ T with all edges doubled

Etour $\leftarrow$ Euler tour in DT

Tour $\leftarrow$ vertices in order of first appearance in ETour

Same analysis as for NA:

$$c_{DT} \leq 2\, c(MST) \leq 2 \cdot c_{OPT}$$

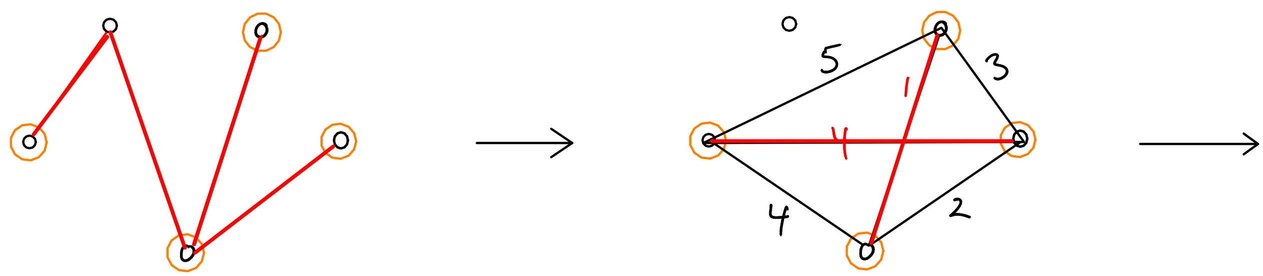
Hence:

Theorem 2.12

Double Tree is a 2-approx. alg

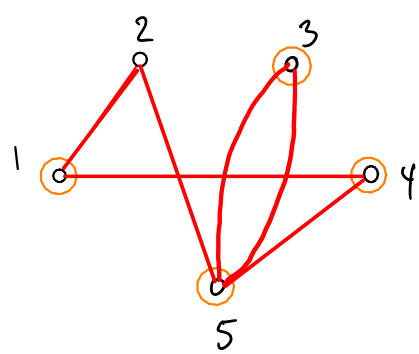
Christofide's Algorithm

Next idea: Not necessary to add n-1 edges to obtain even degree for all vertices.
Instead: add a minimum perfect matching on vertices of odd degree in the MST.



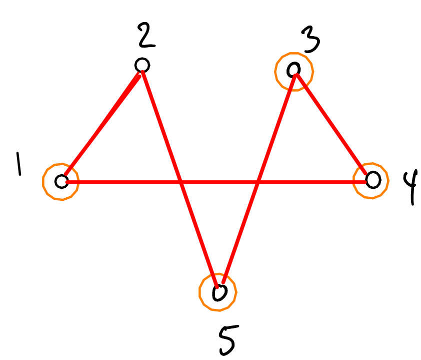
MST
Odd degree

Min. matching



Euler tour : ⟨1, 2, 5, 3, 5, 4, 1⟩

↓ short cutting



TSP tour : ⟨1, 2, 5, 3, 4, 1⟩

Note that it is always possible to find a perfect matching, since there is always an even # odd-degree vertices in T.

Christofide's Algorithm (CA)

$T \leftarrow$ MST

$M \leftarrow$ minimum perfect matching on odd degree vertices in T

ETour $\leftarrow$ Euler tour in the subgraph $(V, E(T) \cup M)$

Tour $\leftarrow$ vertices in order of first appearance in ETour

Theorem 2.13

Christofide's Algorithm is a $3/2$-approx. alg.

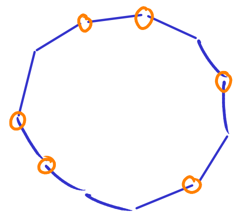
<u>Proof</u>:

By the triangle inequality,

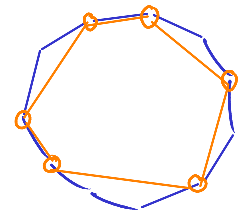
$$\begin{aligned} c_{CA} &\leq c(T) + c(M) \\ &\leq c_{OPT} + c(M), \text{ by Lemma 2.10} \end{aligned}$$

Thus, we just need to prove that

$$c(M) \leq \frac{1}{2} c_{OPT}$$



short cutting $\longrightarrow$



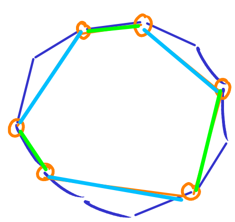
Optimal TSP tour

Odd degree vertices in T

C: cost of orange cycle

$\boxed{C \le c_{OPT}}$, by $\triangle$-ineq.

Since the cycle on the odd degree vertices has an even #edges, it consists of two perfect matchings:



$$C = C + C$$

$$\Downarrow$$

$$\boxed{\min\{C, C\}} \le \frac{1}{2} \cdot C \le \boxed{\frac{1}{2} \cdot c_{OPT}}$$

Since M is a <u>minimum</u> matching on the odd degree vertices,

$$\boxed{c(M) \le \min\{C, C\} \le \frac{1}{2} \cdot c_{OPT}}$$

□

No alg. with an approx. ratio better than $3/2$ is currently known. Moreover:

Theorem 2.14

For $\alpha < \frac{220}{219}$, $\nexists$ α-approx. alg. for Metric TSP

The result of Thm 2.14 is from 2000.
In 2015, the same result was proven for $\alpha < \frac{185}{184}$.