

Chapter 5: Maximum Satisfiability

SAT: For a given boolean formula ϕ in CNF, does there exist a truth assignment satisfying ϕ ?

Conjunctive normal form (CNF): the formula is a conjunction (\wedge) of disjunctions (\vee). Each disjunction is called a **clause**.

Ex: $\phi \equiv \underbrace{(x_1 \vee \bar{x}_2 \vee x_3)}_{\text{clause } C_1} \wedge \underbrace{\bar{x}_3}_{C_2} \wedge \underbrace{(x_1 \vee x_2)}_{C_3}$

positive literal negative literal

x_1, x_2, x_3 are variables

C_j has length/size l_j :

$$l_1 = 3, l_2 = 1, l_3 = 2$$

$x_1 \leftarrow T, x_3 \leftarrow F$ will satisfy ϕ

MAX SAT

Input: Boolean formula ϕ in CNF

with variables x_1, x_2, \dots, x_n

and clauses C_1, C_2, \dots, C_m

Each clause, C_j , has a weight w_j

Output: Truth assignment maximizing the total weight of satisfied clauses

Ex: $(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$
 $w_1=2 \quad w_2=2 \quad w_3=1 \quad w_4=3$

$x_1 \leftarrow T, x_2 \leftarrow F, x_3 \leftarrow T$ satisfies C_1, C_2, C_4
with a total weight of 7.

This is optimal, since we cannot satisfy all clauses:

C_2 requires $x_3 \leftarrow T$

C_3 then requires $x_2 \leftarrow T$

C_1 then requires $x_1 \leftarrow T$

But then C_4 is false.

SAT, and hence, MAX SAT, is NP-hard.

How can we approximate?

Section 5.1: A simple randomized alg.

Consider the following alg:

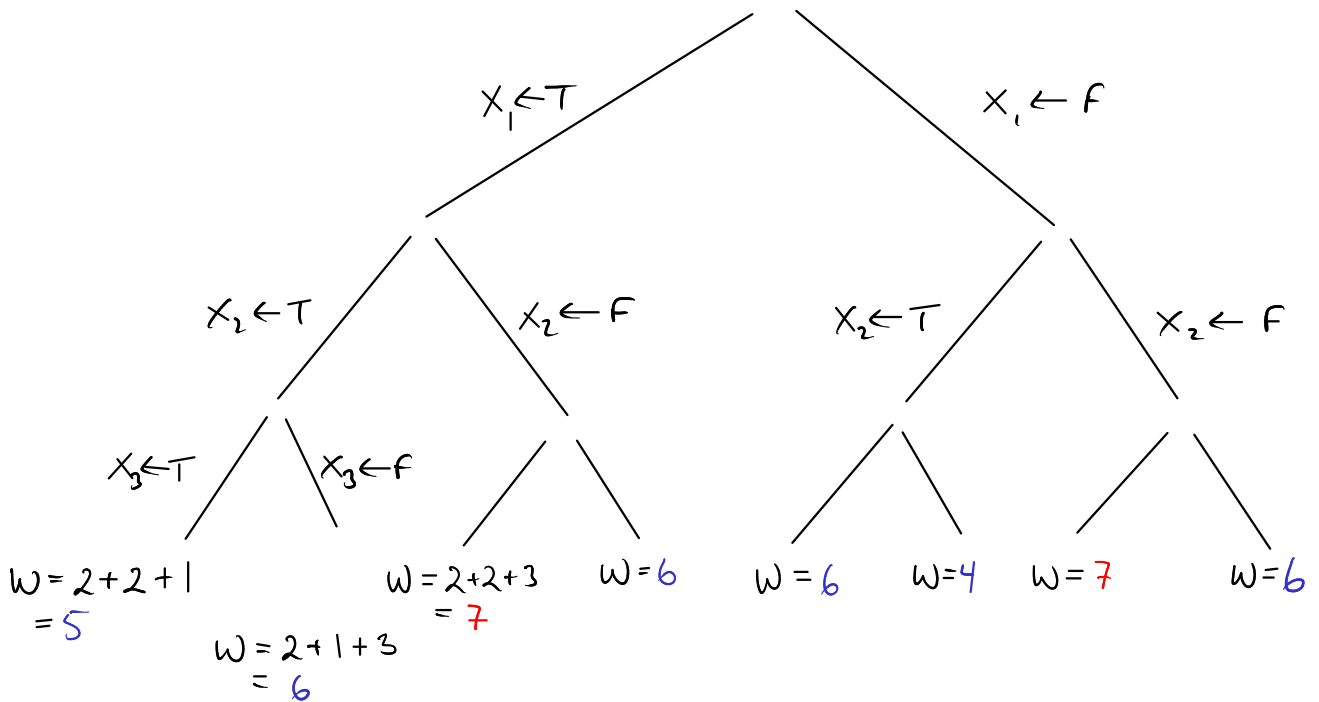
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Rand
For i ← 1 to n
  With prob. 1/2 set xi true
    
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This corresponds to choosing a solution uniformly at random.

Ex: $(x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

₂
₂
₁
₃



Thus, for this example,

$$E[\text{Rand}] = \frac{1}{8}(5 + 6 + 7 + 6 + 6 + 4 + 7 + 6) = 5\frac{7}{8}$$

We don't need to calculate the weight of each possible output...

Instead, we can calculate the exp. weight of each clause:

$$(X_1 \vee \bar{X}_2) \wedge X_3 \wedge (X_2 \vee \bar{X}_3) \wedge (\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3)$$

↑
Satisfied, unless $X_1 \equiv F$ and $X_2 \equiv T$, i.e., satisfied with prob. $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

← satisfied w. prob. $1 - \frac{1}{2} = \frac{1}{2}$

↑
Satisfied w. prob. $1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{8}$

Thus, by linearity of expectation,

$$E[\text{Rand}] = \frac{3}{4} \cdot 2 + \frac{1}{2} \cdot 2 + \frac{3}{4} \cdot 1 + \frac{7}{8} \cdot 3 = 5\frac{7}{8}$$

In general, clause C_j is satisfied with prob. $1 - (\frac{1}{2})^{\ell_j}$.

We let $W \equiv \sum_{j=1}^m w_j$.

Theorem 5.1: Rand is a $\frac{1}{2}$ -approx. alg

Proof:

$$\text{OPT} \leq W$$

By linearity of expectation:

$$\begin{aligned} E[\text{Rand}] &= \sum_{j=1}^m (1 - (\frac{1}{2})^{l_j}) w_j \\ &\geq \frac{1}{2} W, \quad \text{since } l_j \geq 1 \quad \square \end{aligned}$$

In Section 5.1 we got a simple algorithm with a guarantee on the expected performance. We can turn it into a guarantee on the worst-case performance:

Section 5.2: Derandomization

Ex from before:

$$\phi: (x_1 \vee \bar{x}_2) \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

$w_1 = 2 \qquad w_2 = 2 \qquad w_3 = 1 \qquad w_4 = 3$

If we let $x_1 \leftarrow T$, the formula becomes

$$\phi_T \equiv T \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$$

and

$$E[\text{Rand}(\phi_T)] = 2 + \frac{1}{2} \cdot 2 + \frac{3}{4} \cdot 1 + \frac{3}{4} \cdot 3 = 6$$

Or, recalling the probability tree,

$$E[\text{Rand}(\phi_T)] = \frac{1}{4}(5+6+7+6) = 6$$

Similarly, if we let $x_1 \leftarrow F$, the formula becomes

$$\phi_F \equiv \bar{x}_2 \wedge x_3 \wedge (x_2 \vee \bar{x}_3) \wedge T$$

and

$$E[\text{Rand}(\phi_F)] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 + \frac{3}{4} \cdot 1 + 3 = 5\frac{3}{4}$$

$$\text{Or } E[\text{Rand}(\phi_F)] = \frac{1}{4}(6+4+7+6) = 5\frac{3}{4}$$

Note that $E[\text{Rand}]$ is the average of 6 and $5\frac{3}{4}$:

$$E[\text{Rand}] = \frac{1}{2} \cdot E[\text{Rand}(\phi_T)] + \frac{1}{2} \cdot E[\text{Rand}(\phi_F)]$$

Thus,

$$\max \{ E[\text{Rand}(\phi_T)], E[\text{Rand}(\phi_F)] \} \geq E[\text{Rand}]$$

$$\phi: (X_1 \vee \bar{X}_2) \wedge X_3 \wedge (X_2 \vee \bar{X}_3) \wedge (\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3)$$

$$E[\text{Rand}(\phi)] = 5\frac{7}{8}$$

$$X_1 \leftarrow T$$

$$\phi_T: T \wedge X_3 \wedge (X_2 \vee \bar{X}_3) \wedge (\bar{X}_2 \vee \bar{X}_3)$$

$$E[\text{Rand}(\phi_T)] = 6$$

$$X_1 \leftarrow F$$

$$\phi_F: \bar{X}_2 \wedge X_3 \wedge (X_2 \vee \bar{X}_3) \wedge T$$

$$E[\text{Rand}(\phi_F)] = 5\frac{3}{4}$$

$$X_2 \leftarrow T$$

$$\phi_{TT}: T \wedge X_3 \wedge T \wedge \bar{X}_3$$

$$E[\text{Rand}(\phi_{TT})] = 5\frac{1}{2}$$

$$X_2 \leftarrow F$$

$$\phi_{TF}: T \wedge X_3 \wedge \bar{X}_3 \wedge T$$

$$E[\text{Rand}(\phi_{TF})] = 6\frac{1}{2}$$

$$X_3 \leftarrow T$$

$$\phi_{TFT}: T \wedge T \wedge F \wedge T$$

$$\text{Rand}(\phi_{TFT}) = 7$$

$$X_3 \leftarrow F$$

$$\phi_{TFF}: T \wedge F \wedge T \wedge T$$

$$\text{Rand}(\phi_{TFF}) = 6$$

In general:

$$\max \{ E[\text{Rand}(\phi_T)], E[\text{Rand}(\phi_F)] \} \geq E[\text{Rand}] \geq \frac{1}{2} W,$$

The same is true for ϕ_T and ϕ_F :

$$\max \{ E[\text{Rand}(\phi_{TT})], E[\text{Rand}(\phi_{TF})] \} \geq E[\text{Rand}(\phi_T)]$$

and

$$\max \{ E[\text{Rand}(\phi_{FT})], E[\text{Rand}(\phi_{FF})] \} \geq E[\text{Rand}(\phi_F)]$$

Inductively, this proves that the following alg. is a $\frac{1}{2}$ -approx. alg.:

DeRand(ϕ)

For $i \leftarrow 1$ to n

$$\text{If } E[\text{Rand}(\phi_{x_1 \dots x_{i-1} T})] \geq E[\text{Rand}(\phi_{x_1 \dots x_{i-1} F})]$$

$x_i \leftarrow T$

Else

$$x_i \leftarrow F$$

This method of derandomization is sometimes called the method of conditional expectations. (We calculate the conditional exp. of Rand given that $x_i \leftarrow T$ and given that $x_i \leftarrow F$.)

Note that short clauses are "harder" than long clauses:

If all clauses have $l \geq 2$, (Dc)Rand is a $\frac{3}{4}$ -approx. alg.
(In Section 5.3, we will pursue this obs. to obtain a ≈ 0.6 -approx. alg.)

If all clauses have $l \geq 3$, (Dc)Rand is a $\frac{7}{8}$ -approx. alg.

In some sense, this is optimal:

MAX ESSAT: The special case of MAX SAT where $l=3$ for all clauses.

Theorem 5.2:

$\exists \epsilon > 0 : \exists (\frac{7}{8} + \epsilon)$ -approx alg for MAX ESSAT $\Rightarrow P = NP$

Section 5.3: A biased rand. alg.

Since **unit clauses** (clauses of exactly one literal) are the "hardest", we should focus on these to obtain a better approx. ratio.

For each i , $1 \leq i \leq n$, we define:

$$u_i \equiv \begin{cases} \text{weight of unit clause } x_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

$$v_i \equiv \begin{cases} \text{weight of unit clause } \bar{x}_i, & \text{if it exists} \\ 0, & \text{otherwise} \end{cases}$$

Idea: If $u_i \geq v_i$, set x_i true with prob. $> \frac{1}{2}$, and vice versa.

For ease of presentation, assume that

$$u_i \geq v_i, \quad 1 \leq i \leq n$$

Why is this not a restriction?

Thus, each variable will be set true with prob. $> \frac{1}{2}$:

For any $p > \frac{1}{2}$, we define the following alg:

Rand_p

for $i \leftarrow 1$ to n
 With prob. p set x_i true

What is an optimal value of p ?

Lemma 5.4

For any clause C_j which does not consist of one negated variable,
Rand_p satisfies C_j with prob $\geq \min\{p, 1-p^2\}$

Proof:

If $l_j = 1$, C_j consists of one unnegated variable.
In this case, C_j is satisfied with prob. p .

If $l_j = 2$, the worst case is if both literals are negated variables, since $p > \frac{1}{2}$. Thus, in this case, C_j is satisfied with prob. $\geq 1-p^2$.

If $l_j \geq 3$, the prob. of C_j being satisfied is at least the worst-case prob. for $l_j = 2$. □

$$\text{Lemma 5.6: } \text{OPT} \leq W - \sum_{i=1}^n \sigma_i$$

Proof:

By assumption, $u_i \geq \sigma_i$, for all i .

Thus, if $\sigma_i > 0$, there is both an x_i - and an \bar{x}_i -clause. Both clauses cannot be satisfied.

Thus, for each $\sigma_i > 0$, there is an unsatisfied clause of weight $\geq \min\{u_i, \sigma_i\} = \sigma_i$. □

We can obtain an alg. with approx. ratio $\frac{1}{2}(\sqrt{5}-1) \approx 0,618$:

Theorem 5.7:

For $\rho = \frac{1}{2}(\sqrt{5}-1)$, Rand_ρ is a ρ -approx. alg.

Proof:

By Lemma 5.4,

Total weight of clauses that
are not negated unit clauses

$$E[\text{Rand}_p] \geq \min\{\rho, 1-\rho^2\} \cdot \left(W - \sum_{i=1}^n \sigma_i\right) \\ = \rho \left(W - \sum_{i=1}^n \sigma_i\right), \text{ for } \rho = \frac{1}{2}(\sqrt{5}-1):$$

$$1 - \left(\frac{1}{2}(\sqrt{5}-1)\right)^2 = 1 - \frac{1}{4}(5+1-2\sqrt{5}) = 1 - \frac{3}{2} + \frac{1}{2}\sqrt{5} \\ = \frac{1}{2}(\sqrt{5}-1)$$

By Lemma 5.6, $OPT \leq W - \sum_{i=1}^n \sigma_i$.

Hence, for $\rho = \frac{1}{2}(\sqrt{5}-1)$, $\frac{E[\text{Rand}_p]}{OPT} \geq \rho$ \square

Note that

Rand_p can be derandomized exactly like Rand