Section 5.4: Randomized rounding

In Section 5.3 we saw that biasing the prob. of setting each variable true resulted in a better approx. guarantee.
The approximation ratio can be further improved by allowing a different bias for each variable.
We will develop an LP-formulation of the problem.

For each clause, C_j, we define:
P_j : the set of indices of variables that occur positively in C_j
N_j : ———— " ———— negatively —"—

Then, C_j can be written as $\bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$

Ex: $\cdots \wedge \overbrace{(x_1 \vee \bar{x}_2 \vee \bar{x}_4 \vee x_7)}^{C_j} \wedge \cdots$

$P_j = \{1,7\}$, $N_j = \{2,4\}$

$$((x_1 \vee x_7) \vee (\bar{x}_2 \vee \bar{x}_4))$$

If $y_i = 0 \iff x_i \equiv F$ and $y_i = 1 \iff x_i \equiv T$,
then C_j is true, iff

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq 1$$

This leads to the following IP-formulation:

IP_ϕ:

$$\max \sum_{j=1}^{m} z_j w_j$$

subject to

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j , \quad 1 \leq j \leq m$$

$$y_i \in \{0,1\}, \quad 1 \leq i \leq n$$

$$z_j \in \{0,1\}, \quad 1 \leq j \leq m$$

Let LP_ϕ be the LP-relaxation of IP_ϕ, i.e.,

LP_ϕ:

$$\max \sum_{j=1}^{m} z_j w_j$$

subject to

$$\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j , \quad 1 \leq j \leq m$$

$$0 \leq y_i \leq 1 , \quad 1 \leq i \leq n$$

$$0 \leq z_j \leq 1 , \quad 1 \leq j \leq m$$

Clearly, $Z^*_{LP_\phi} \geq Z^*_{IP_\phi} = OPT$

value of opt. sol. to LP_ϕ

value of opt. sol. to IP_ϕ

value of opt. sol. to corresponding MAXSAT problem

RandRounding(ϕ)

$(\vec{y^*}, \vec{z^*}) \leftarrow$ opt. sol. to LP_ϕ
For $i \leftarrow 1$ to n
    Set x_i true with prob. y_i^*

(i.e., $Z^*_{LP_\phi} = \sum_{j=1}^{m} z_j^* w_j$)

The approx. ratio of RandRounding is at least $1-\frac{1}{e} \approx 0.632$.
For proving this, we will use the following two facts:

Fact 5.8 (Arithmetic-geometric mean inequality):
For any $a_1, a_2, \dots, a_k \geq 0$,

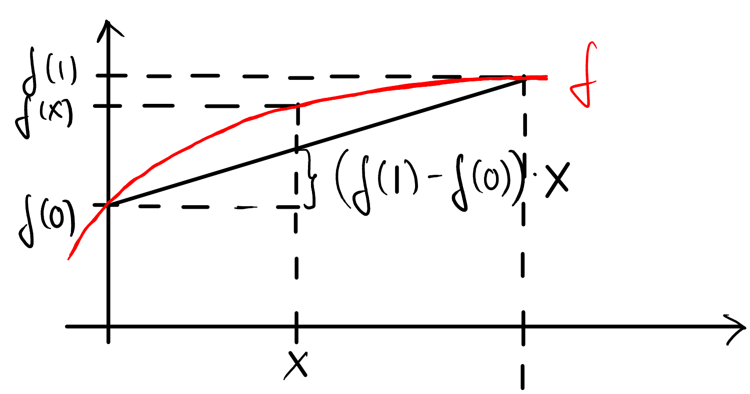
$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} a_i$$

$\Downarrow$

$$\prod_{i=1}^{k} a_i \leq \left(\frac{1}{k}\sum_{i=1}^{k} a_i\right)^k$$

A function f is concave on an interval I,
if $f''(x) \leq 0$ for any $x \in I$. (the slope is nonincreasing)

Fact 5.9:
f is concave on $[0,1]$
$\Downarrow$ $\forall x \in [0,1]: f(x) \geq (f(1)-f(0))x + f(0)$



$\boxed{\text{Theorem 5.10: RandRounding is a } (1-\frac{1}{e})\text{-approx. alg}}$

<u>Proof</u>:

For $1 \le j \le m$, let p_j be the probability that C_j is satisfied, and let $\overline{p_j} = 1 - p_j$.

Our goal is to show that $p_j \geq (1-\frac{1}{e}) z_j^*$.

This will establish the approx. factor, since $OPT \le \sum_{j=1}^{m} z_j^* w_j$

$$\overline{p_j} = \prod_{i \in P_j} (1-y_i^*) \prod_{i \in N_j} y_i^*$$

$$\le \left(\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1-y_i^*) + \sum_{i \in N_j} y_i^* \right) \right)^{\ell_j}, \text{ by Fact 5.8}$$

$$= \left(\frac{1}{\ell_j} \left(|P_j| - \sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - 1 + y_i^*) \right) \right)^{\ell_j}$$

$$= \left(\frac{1}{\ell_j} \left(|P_j| - \sum_{i \in P_j} y_i^* + |N_j| - \sum_{i \in N_j} (1-y_i^*) \right) \right)^{\ell_j}$$

$$= \left(1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1-y_i^*) \right) \right)^{\ell_j}, \text{ since } |P_j| + |N_j| = \ell_j$$

$$\le \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j}, \text{ since } (\vec{y}^*, \vec{z}^*) \text{ is a solution to } LP_\phi$$

Thus,

$$p_j \geq 1 - \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j} \equiv f(z_j^*)$$

which is a concave function of z_j^*:

$$f'(z_j^*) = -\ell_j \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 1} \cdot \left(-\frac{1}{\ell_j}\right) = \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 1}$$

$$f''(z_j^*) = (\ell_j - 1)\left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 2} \cdot \left(-\frac{1}{\ell_j}\right) = \underbrace{\left(\frac{1}{\ell_j} - 1\right)}_{\le 0} \underbrace{\left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j - 2}}_{\geq 0}$$

$$\le 0$$

Note that

$$f(0) = 1 - \left(1 - \frac{0}{\ell_j}\right)^{\ell_j} = 1 - 1 = 0$$

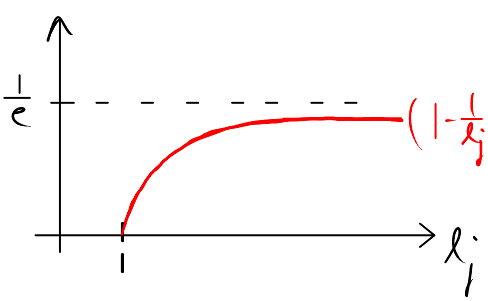
$$f(1) = 1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}$$

Thus,

$$\begin{aligned} p_j &\geq f(z_j^*) \\ &\geq \big(f(1) - f(0)\big)\, z_j^* + f(0), \quad \text{by Fact 5.9} \\ &\geq \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^* \end{aligned}$$

Hence,

$$\begin{aligned} E[\text{Rand Rounding}] &= \sum_{j=1}^{m} p_j w_j \\ &\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^* w_j \\ &\geq \left(1 - \frac{1}{e}\right) \cdot \underbrace{\sum_{j=1}^{m} z_j^* w_j}_{= Z^*_{LP_\phi} \geq OPT} \end{aligned}$$



$\square$

Note that

$$\boxed{\text{Rand Rounding can be derandomized exactly like Rand and Rand}_p}$$

Section 5.5 : Choosing the better of two solutions

Combining the alg.s of Sections 5.1 and 5.4 gives a better approx. factor than using any one of them separately. This is because they have different worst-case inputs:

Rand satisfies clause C_j with prob. $p_R = 1-(\frac{1}{2})^{l_j}$.

RandRounding satisfies C_j with prob. $p_{RR} \geq (1-(1-\frac{1}{l_j})^{l_j}) z_j^*$.

While p_R increases with l_j, the lower bound on p_{RR} decreases with l_j.

BestOfTwo(ϕ)

$\vec{x}_R \leftarrow$ Rand(ϕ)
$\vec{x}_{RR} \leftarrow$ RandRounding(ϕ)
If $w(\phi, \vec{x}_R) \geq w(\phi, \vec{x}_{RR})$
    Return $\vec{x}_R$
Else
    Return $\vec{x}_{RR}$

Note that

BestOfTwo is derandomized by using the derandomized versions of Rand and RandRounding.

$\boxed{\text{Theorem 5.11: BestOfTwo is a } 3/4\text{-approx. alg.}}$

<u>Proof:</u>

$$\begin{aligned}
E[\text{BestOfTwo}(\phi)] &= E[\max\{\text{Rand}(\phi), \text{RandRounding}(\phi)\}] \\
&\geq E\left[\tfrac{1}{2}\text{Rand}(\phi) + \tfrac{1}{2}\text{RandRounding}(\phi)\right] \\
&= \tfrac{1}{2}E[\text{Rand}(\phi)] + \tfrac{1}{2}E[\text{RandRounding}(\phi)], \quad \text{by lin. of exp.} \\
&\geq \frac{1}{2}\sum_{j=1}^{m}\left(1-2^{-\ell_j}\right)w_j + \frac{1}{2}\sum_{j=1}^{m}\left(1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right)z_j^* w_j \\
&\geq \sum_{j=1}^{m} z_j^* w_j \cdot \underbrace{\frac{1}{2}\left(1-2^{-\ell_j} + 1-\left(1-\frac{1}{\ell_j}\right)^{\ell_j}\right)}_{\equiv\, \rho_j}, \quad \text{since } z_j^* \leq 1.
\end{aligned}$$

For $\ell_j = 1$, $\rho_j = \frac{1}{2}\left(1-\frac{1}{2}+1-0\right) = \frac{3}{4}$

For $\ell_j = 2$, $\rho_j = \frac{1}{2}\left(1-\frac{1}{4}+1-\left(1-\frac{1}{2}\right)^2\right) = \frac{3}{4}$

For $\ell_j \geq 3$, $\rho_j \geq \frac{1}{2}\left(1-\frac{1}{8}+1-\frac{1}{e}\right) > \frac{3}{4}$

Hence,

$$E[\text{BestOfTwo}] \geq \sum_{j=1}^{m} z_j^* w_j \cdot \frac{3}{4} \geq \frac{3}{4}\cdot \text{OPT} \qquad \square$$