Section 5.6: Nonlinear randomized rounding

RandRounding$_f(\phi)$

$(\vec{y^*}, \vec{z^*}) \leftarrow$ opt. sol. to LP_ϕ
For $i \leftarrow 1$ to n
  Set x_i true with prob. $f(y_i^*)$

Theorem 5.12

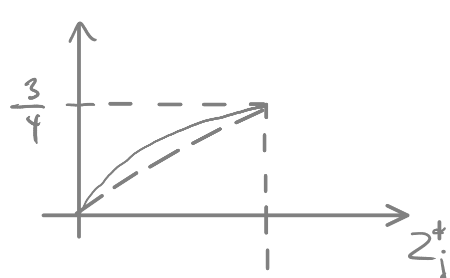
RandRounding$_f$ is a $3/4$-approx. alg., if $1-4^{-x} \leq f(x) \leq 4^{x-1}$

<u>Proof</u>:

Prob. that C_j is <u>not</u> satisfied:

$$\bar{p}_j = \prod_{i \in P_j} (1-f(y_i^*)) \prod_{i \in N_j} f(y_i^*)$$
$$\leq \prod_{i \in P_j} 4^{-y_i^*} \prod_{i \in N_j} 4^{y_i^*-1}$$
$$= 4^{-\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} 1-y_i^*\right)}$$
$$\leq 4^{-z_j^*}$$

Prob. that C_j <u>is</u> satisfied:



$$p_j = 1-\bar{p}_j \geq 1 - 4^{-z_j^*}$$
$$\geq 0 + \left(\tfrac{3}{4}-0\right) z_j^*, \text{ by Fact 5.9}$$
$$= \tfrac{3}{4} z_j^*$$

$\square$

Ex: $\phi \equiv (x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

$w_1 = w_2 = w_3 = w_4 = 1$

$OPT = 3$

$y_1 = y_2 = \frac{1}{2} \Rightarrow Z = 4$

Hence, the integrality gap for the IP problem for MaxSat is

$$\min_{\psi} \left\{ \frac{Z^*_{IP_\psi}}{Z^*_{LP_\psi}} \right\} \leq \frac{Z^*_{IP_\phi}}{Z^*_{LP_\phi}} = \frac{3}{4}$$

On the other hand, the proof that $Rand_f$ is a 3/4-approx. alg. shows that for any instance ψ of MaxSat, $Rand_f(\psi) \geq \frac{3}{4} Z^*_{LP_\psi}$. Hence,

$$\frac{Z^*_{IP_\psi}}{Z^*_{LP_\psi}} \geq \frac{Rand_f(\psi)}{Z^*_{LP_\psi}} \geq \frac{3}{4}$$

Hence, the integrality gap is exactly $\frac{3}{4}$.

The upper bound of $\frac{3}{4}$ on the integrality gap shows that we cannot prove an approx. factor better than $\frac{3}{4}$, if the approximation guarantee is based on a comparison to Z^*_{LP}:

$$\min_{\psi} \left\{ \frac{ALG(\psi)}{Z^*_{LP_\psi}} \right\} \leq \min \left\{ \frac{OPT(\psi)}{Z^*_{LP_\psi}} \right\} \leq \frac{3}{4}$$

$\boxed{\text{Set Cover}}$

Techniques: (with Set Cover as an example)

- Solve LP and round solution (Sec. 1.3 + 1.7)
- Primal-dual alg.: combinatorial alg. based on LP formulation (Sec. 1.4 + 1.5)
- Greedy alg. (Sec. 1.6)

Section 1.2: Set Cover as an LP

Set Cover

Input:

$E = \{e_1, e_2, \ldots, e_n\}$

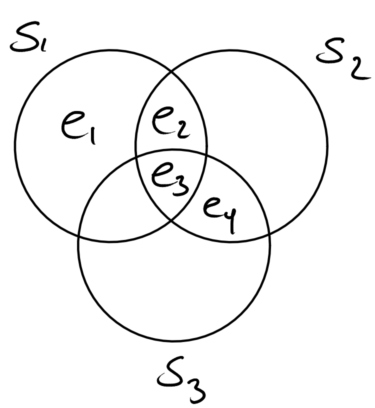
$\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, where

$S_j \subseteq E$ has weight w_j.

Objective: Find a cheapest possible subset of $\mathcal{S}$ covering all elements

OPT: value (total weight) of optimum solution

Ex:



$w_1 = 1$

$w_2 = 2$

$w_3 = 3$

$\{S_1, S_2\}$ is a sol. of total weight 3.

This is optimal, so $OPT = 3$ for this instance of Set Cover.

To cover e_1, we need S_1

—"— e_2 —"— S_1 or S_2

—"— e_3 —"— S_1, S_2 or S_3

—"— e_4 —"— S_2 or S_3

IP-formulation:

$$\min \quad x_1 w_1 + x_2 w_2 + x_3 w_3$$

$$\begin{aligned}
\text{s.t.} \quad & x_1 \geq 1 \\
& x_1 + x_2 \geq 1 \\
& x_1 + x_2 + x_3 \geq 1 \\
& x_2 + x_3 \geq 1 \\
& x_1, x_2, x_3 \in \{0,1\}
\end{aligned}$$

More generally:

IP for Set Cover

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \quad \sum_{j : e_i \in S_j} x_j \geq 1 \quad , \qquad i = 1, 2, \ldots, n$$

$$x_j \in \{0, 1\} \quad , \qquad j = 1, 2, \ldots, m$$

Z^*_{IP} : optimum solution value, i.e., $Z^*_{IP} = OPT$

LP- relaxation

$$\min \sum_{j=1}^{m} x_j w_j$$

$$\text{s.t.} \quad \sum_{j : e_i \in S_j} x_j \geq 1 \quad , \qquad i = 1, 2, \ldots, n$$

$$0 \leq x_j \leq 1 \quad , \qquad j = 1, 2, \ldots, m$$

redundant

Z^*_{LP} : Optimum solution value

Note that $\boxed{Z^*_{LP} \leq Z^*_{IP} = OPT}$

Section 1.3: A deterministic rounding algo.

The frequency of an element e is the #sets containing e:

$$f_e = |\{S \in \mathcal{S} \mid e \in S\}|$$

The frequency of an instance of Set Cover:

$$f = \max_{e \in E} \{f_e\}$$

Alg. 1 for Set Cover: LP-rounding

$\vec{x}^* \leftarrow$ opt. sol. to LP

$I \leftarrow \{ j \mid x_j^* \geq \frac{1}{f} \}$

We prove that Alg 1 produces a set cover (Lemma 1.5) of total weight $\leq f \cdot OPT$ (Thm 1.6)

$\boxed{\text{Lemma 1.5}}$

$\boxed{\{ S_j \mid j \in I \} \text{ is a set cover}}$

Proof:

For each $e_i \in E$, $\sum_{j : e_i \in S_j} x_j \geq 1$.

Since $\sum_{j : e_i \in S_j} x_j$ has at most f terms, at least one of the terms is at least $\frac{1}{f}$.

Thus, there is a set S_j s.t.
$e_i \in S_j$ and $x_j \geq \frac{1}{f}$.
This j is included in I $\square$

$\boxed{\text{Thm 1.6}}$

$\boxed{\text{Alg. 1 is an } f\text{-approx. algo. for Set Cover.}}$

Proof:

Correct by Lemma 1.5
Poly, since LP-solving is poly.
Approx. factor f:
Each x_j is rounded up to 1, only if it is already at least $\frac{1}{f}$.
Thus, each x_j is multiplied by at most f, i.e.,

$$\sum_{j \in I} w_j \leq \sum_{j \in I} f \cdot x_j^* \cdot w_j \leq \sum_{j=1}^{m} f \cdot x_j^* \cdot w_j = f \cdot Z_{LP}^* \leq f \cdot OPT$$

$\square$

The Vertex Cover problem is a special case of Set Cover:

Vertex Cover

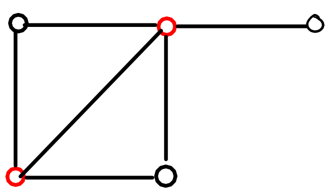
Input:

$G = (V, E)$

Objective:

Find a min. card. vertex set $C \subseteq V$ s.t. each edge $e \in E$ has at least one endpoint in C.

Ex:



With $\mathcal{S} = V$ and $\mathcal{E} = E$, Alg. 1 is a 2-approx. alg. for Vertex Cover.

One of the exercises for Tuesday: Write down LP for Vertex Cover.

Section 1.4: The dual LP

What is a dual?

Ex:

P

$$\begin{aligned}
\min \quad & 7x_1 + x_2 + 5x_3 \\
\text{s.t.} \quad & x_1 - x_2 + 3x_3 \geqslant 10 \\
& 5x_1 + 2x_2 - x_3 \geqslant 6 \\
& x_1, x_2, x_3 \geqslant 0
\end{aligned}$$

Primal

$$7x_1 + x_2 + 5x_3 \geqslant x_1 - x_2 + 3x_3 \geqslant 10$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geqslant x_1 - x_2 + 3x_3 + 5x_1 + 2x_2 - x_3 \\
&\geqslant 10 + 6 = 16
\end{aligned}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geqslant 2(x_1 - x_2 + 3x_3) + 5x_1 + 2x_2 - x_3 \\
&\geqslant 2 \cdot 10 + 6 = 26
\end{aligned}$$

To find a largest possible lower bound on $7x_1 + x_2 + 5x_3$, we should determine y_1 and y_2 maximizing $10y_1 + 6y_2$, under the constraints that

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\overset{(*)}{\geqslant} y_1 \overbrace{(x_1 - x_2 + 3x_3)}^{\geqslant 10} + y_2 \overbrace{(5x_1 + 2x_2 - x_3)}^{\geqslant 6} \\
&= \underbrace{(y_1 + 5y_2)}_{\leqslant 7} x_1 + \underbrace{(-y_1 + 2y_2)}_{\leqslant 1} x_2 + \underbrace{(3y_1 - y_2)}_{\leqslant 5} x_3
\end{aligned}$$

and $\underbrace{y_1, y_2, y_3 \geqslant 0}_{\text{otherwise}}$

„$\geqslant$" becomes „$\leqslant$"

necessary to satisfy (*)

Thus, we arrive at the following problem:

D

$$\begin{aligned}
\max \quad & 10y_1 + 6y_2 \\
\text{s.t.} \quad & y_1 + 5y_2 \leq 7 \\
& -y_1 + 2y_2 \leq 1 \\
& 3y_1 - y_2 \leq 5 \\
& y_1, y_2 \geq 0
\end{aligned}$$

Dual

<u>In general:</u>

Primal:

$$\begin{aligned}
\min \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\
\text{s.t.} \quad & a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \geq b_i, \quad i = 1, 2, \dots, m \\
& x_j \geq 0, \quad j = 1, 2, \dots, n
\end{aligned}$$

Dual:

$$\begin{aligned}
\max \quad & b_1 y_1 + b_2 y_2 + \dots + b_m y_m \\
\text{s.t.} \quad & a_{1j} y_1 + a_{2j} y_2 + \dots + a_{mj} y_m \leq c_j, \quad j = 1, 2, \dots, n \\
& y_i \geq 0, \quad i = 1, 2, \dots, m
\end{aligned}$$