DM865 – Spring 2020
Heuristics and Approximation Algorithms

Complexity

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

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Complexity Hierarchy

Reduction

A search problem Π' is (polynomially) reducible to a search problem Π ($\Pi' \longrightarrow \Pi$) if there exists an algorithm $\mathcal{A}$ that solves Π' by using a hypothetical subroutine $\mathcal{S}$ for Π and except for $\mathcal{S}$ everything runs in polynomial time. [Garey and Johnson, 1979]

NP-hard

A search problem Π is NP-hard if

1. it is in NP
2. there exists some NP-complete problem Π' that reduces to Π

In scheduling, complexity hierarchies describe relationships between different problems.

Ex: $1||\sum C_j \longrightarrow 1||\sum w_j C_j$

Interest in characterizing the borderline: polynomial vs NP-hard problems

Problems Involving Numbers

Partition

- **Input:** finite set A and a size $s(a) \in \mathbf{Z}^+$ for each $a \in A$
- **Question:** is there a subset $A' \subseteq A$ such that

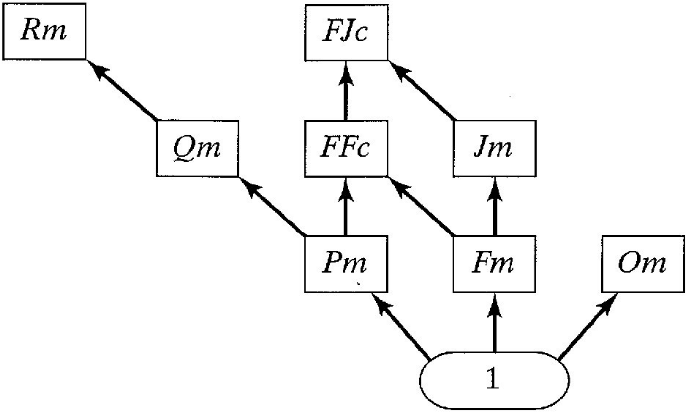
$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)?$$

3-Partition

- **Input:** set A of $3m$ elements, a bound $B \in \mathbf{Z}^+$, and a size $s(a) \in \mathbf{Z}^+$ for each $a \in A$ such that $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$
- **Question:** can A be partitioned into m disjoint sets $A_1, \ldots, A_m$ such that for $1 \leq i \leq m$, $\sum_{a \in A_i} s(a) = B$ (note that each A_i must therefore contain exactly three elements from A)?

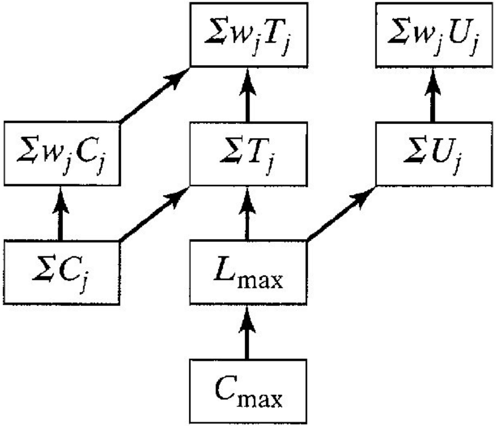
Complexity Hierarchy

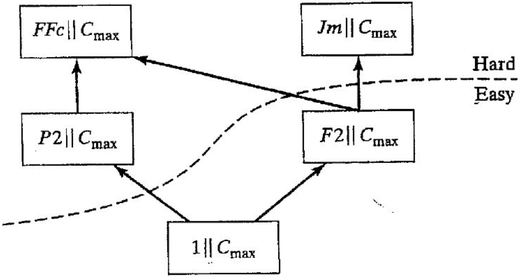
Elementary reductions for machine environment



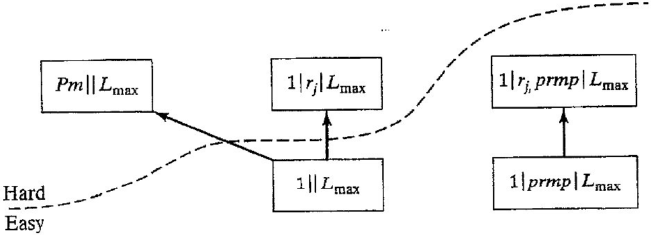
Complexity Hierarchy

Elementary reductions for regular objective functions





$FFc||C_{max}$
$Jm||C_{max}$
Hard
Easy
$P2||C_{max}$
$F2||C_{max}$
$1||C_{max}$



$Pm||L_{max}$
$1|r_j|L_{max}$
$1|r_j, prmp|L_{max}$
Hard
Easy
$1||L_{max}$
$1|prmp|L_{max}$

Polynomial time solvable problems

SINGLE MACHINE	PARALLEL MACHINES	SHOPS
$1 \mid r_j, p_j = 1, prec \mid \sum C_j$	$P2 \mid p_j = 1, prec \mid L_{\max}$	$O2 \mid\mid C_{\max}$
$1 \mid r_j, prmp \mid \sum C_j$	$P2 \mid p_j = 1, prec \mid \sum C_j$	
$1 \mid tree \mid \sum w_j C_j$		$Om \mid r_j, prmp \mid L_{\max}$
	$Pm \mid p_j = 1, tree \mid C_{\max}$	
$1 \mid prec \mid L_{\max}$	$Pm \mid prmp, tree \mid C_{\max}$	$F2 \mid block \mid C_{\max}$
$1 \mid r_j, prmp, prec \mid L_{\max}$	$Pm \mid p_j = 1, outtree \mid \sum C_j$	$F2 \mid nwt \mid C_{\max}$
	$Pm \mid p_j = 1, intree \mid L_{\max}$	
$1 \mid\mid \sum U_j$	$Pm \mid prmp, intree \mid L_{\max}$	$Fm \mid p_{ij} = p_j \mid \sum C_j$
$1 \mid r_j, prmp \mid \sum U_j$		$Fm \mid p_{ij} = p_j \mid L_{\max}$
$1 \mid r_j, p_j = 1 \mid \sum w_j U_j$	$Q2 \mid prmp, prec \mid C_{\max}$	$Fm \mid p_{ij} = p_j \mid \sum U_j$
	$Q2 \mid r_j, prmp, prec \mid L_{\max}$	
$1 \mid r_j, p_j = 1 \mid \sum w_j T_j$		$J2 \mid\mid C_{\max}$
	$Qm \mid r_j, p_j = 1 \mid C_{\max}$	
	$Qm \mid p_j = 1, M_j \mid C_{\max}$	
	$Qm \mid r_j, p_j = 1 \mid \sum C_j$	
	$Qm \mid prmp \mid \sum C_j$	
	$Qm \mid p_j = 1 \mid \sum w_j C_j$	
	$Qm \mid p_j = 1 \mid L_{\max}$	
	$Qm \mid prmp \mid \sum U_j$	
	$Qm \mid p_j = 1 \mid \sum w_j U_j$	
	$Qm \mid p_j = 1 \mid \sum w_j T_j$	
	$Rm \mid\mid \sum C_j$	
	$Rm \mid r_j, prmp \mid L_{\max}$	

NP-hard problems in the ordinary sense

SINGLE MACHINE	PARALLEL MACHINES	SHOPS
$1 \mid\mid \sum w_jU_j$ (*) $1 \mid r_j, prmp \mid \sum w_jU_j$ (*) $1 \mid\mid \sum T_j$ (*)	$P2 \mid\mid C_{\max}$ (*) $P2 \mid r_j, prmp \mid \sum C_j$ $P2 \mid\mid \sum w_jC_j$ (*) $P2 \mid r_j, prmp \mid \sum U_j$ $Pm \mid prmp \mid \sum w_jC_j$ $Qm \mid\mid \sum w_jC_j$ (*) $Rm \mid r_j \mid C_{\max}$ (*) $Rm \mid\mid \sum w_jU_j$ (*) $Rm \mid prmp \mid \sum w_jU_j$	$O2 \mid prmp \mid \sum C_j$ $O3 \mid\mid C_{\max}$ $O3 \mid prmp \mid \sum w_jU_j$

Strongly NP-hard problems

SINGLE MACHINE	PARALLEL MACHINES	SHOPS
$1 \mid s_{jk} \mid C_{\max}$	$P2 \mid chains \mid C_{\max}$	$F2 \mid r_j \mid C_{\max}$
	$P2 \mid chains \mid \sum C_j$	$F2 \mid r_j, prmp \mid C_{\max}$
$1 \mid r_j \mid \sum C_j$	$P2 \mid prmp, chains \mid \sum C_j$	$F2 \mid\mid \sum C_j$
$1 \mid prec \mid \sum C_j$	$P2 \mid p_j = 1, tree \mid \sum w_j C_j$	$F2 \mid prmp \mid \sum C_j$
$1 \mid r_j, prmp, tree \mid \sum C_j$		$F2 \mid\mid L_{\max}$
$1 \mid r_j, prmp \mid \sum w_j C_j$	$R2 \mid prmp, chains \mid C_{\max}$	$F2 \mid prmp \mid L_{\max}$
$1 \mid r_j, p_j = 1, tree \mid \sum w_j C_j$		
$1 \mid p_j = 1, prec \mid \sum w_j C_j$		$F3 \mid\mid C_{\max}$
		$F3 \mid prmp \mid C_{\max}$
$1 \mid r_j \mid L_{\max}$		$F3 \mid nwt \mid C_{\max}$
$1 \mid r_j \mid \sum U_j$		$O2 \mid r_j \mid C_{\max}$
$1 \mid p_j = 1, chains \mid \sum U_j$		$O2 \mid\mid \sum C_j$
		$O2 \mid prmp \mid \sum w_j C_j$
$1 \mid r_j \mid \sum T_j$		$O2 \mid\mid L_{\max}$
$1 \mid p_j = 1, chains \mid \sum T_j$		
$1 \mid\mid \sum w_j T_j$		$O3 \mid prmp \mid \sum C_j$
		$J2 \mid rcrc \mid C_{\max}$
		$J3 \mid p_{ij} = 1, rcrc \mid C_{\max}$

Web Archive

Complexity results for scheduling problems
by Peter Brucker and Sigrid Knust
http://www.informatik.uni-osnabrueck.de/knust/class/