DM545/DM871

Linear and Integer Programming

Lecture 9

IP Modeling
Formulations, Relaxations

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Outline

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Uncapacited Facility Location (UFL)

Given:

- depots $N = \{1, \ldots, n\}$
- clients $M = \{1, \ldots, m\}$
- f_j fixed cost to use depot j
- transport cost for all orders c_{ij}

Task: Which depots to open and which depots serve which client

Variables: $y_j = \begin{cases} 1 & \text{if depot opened} \\ 0 & \text{otherwise} \end{cases}$, x_{ij} fraction of demand of i satisfied by j

Objective:

$$\min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

Constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, m$$

$$\sum_{i \in M} x_{ij} \leq m y_j \qquad \forall j \in N$$

Outline

Good and Ideal Formulations

Definition (Formulation)

A polyhedron $P \subseteq \mathbb{R}^{n+p}$ is a formulation for a set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$ if and only if $X = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$

That is, if it does not leave out any of the solutions of the feasible region X.

There are infinite formulations

Definition (Convex Hull)

Given a set $X \subseteq \mathbb{Z}^n$ the convex hull of X is defined as:

$$\text{conv}(X) = \Big\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{t} \lambda_i \mathbf{x}^i, \quad \sum_{i=1}^{t} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{for } i = 1, \ldots, t, \\ \text{for all finite subsets } \{\mathbf{x}^1, \ldots, \mathbf{x}^t\} \text{ of } X \Big\}$$

Proposition

$conv(X)$ is a polyhedron (ie, representable as $A\mathbf{x} \leq \mathbf{b}$)

Proposition

Extreme points of $conv(X)$ all lie in X

Hence:

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\} \equiv \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \text{conv}(X)\}$$

However it might require exponential number of inequalities to describe $\text{conv}(X)$
What makes a formulation better than another?

$$X \subseteq \text{conv}(X) \subseteq P_2 \subset P_1$$

$$P_2 \text{ is better than } P_1$$

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_2 is a better formulation than P_1 if $P_2 \subset P_1$

Example

$P_1 = $ UFL with $\sum_{i \in M} x_{ij} \leq m y_j \quad \forall j \in N$
$P_2 = $ UFL with $x_{ij} \leq y_j \quad \forall i \in M, j \in N$

$$P_2 \subset P_1$$

- $P_2 \subseteq P_1$ because summing $x_{ij} \leq y_j$ over $i \in M$ we obtain $\sum_{i \in M} x_{ij} \leq m y_j$
- $P_2 \subset P_1$ because there exists a point in P_1 but not in P_2: $m = 6 = 3 \cdot 2 = k \cdot n$

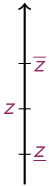
$$x_{10} = 1,\ x_{20} = 1,\ x_{30} = 1, \qquad \sum_i x_{i0} \leq 6 y_0 \quad y_0 = 1/2$$
$$x_{41} = 1,\ x_{51} = 1,\ x_{61} = 1 \qquad \sum_i x_{i1} \leq 6 y_1 \quad y_1 = 1/2$$

Outline

Optimality and Relaxation

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

How can we prove that $\mathbf{x}^*$ is optimal?
$\overline{z}$ is UB
$\underline{z}$ is LB
stop when $\overline{z} - \underline{z} \leq \epsilon$



- Primal bounds (here lower bounds): every feasible solution gives a primal bound
  may be easy or hard to find, heuristics
- Dual bounds (here upper bounds): Relaxations

Optimality gap (SCIP):

- If primal and dual bound have opposite signs, the gap is "Infinity".
- If primal and dual bound have the same sign, the gap is

$$\frac{|pb - db|}{\min(|pb|, |db|)}$$

decreases monotonously during the solving process.

Proposition

$$
\begin{array}{l}
(RP)\ z^R = \max\{f(\mathbf{x}) : \mathbf{x} \in T \subseteq \mathbb{R}^n\} \textit{ is a relaxation of} \\
(IP)\ z \ \ \ = \max\{c(\mathbf{x}) : \mathbf{x} \in X \subseteq \mathbb{R}^n\} \textit{ if :}
\end{array}
$$

(i) $X \subseteq T$ *or*

(ii) $f(\mathbf{x}) \geq c(\mathbf{x})\ \forall \mathbf{x} \in X$

In other terms:

$$
\max_{\mathbf{x} \in T} f(\mathbf{x}) \geq \left\{ \begin{array}{l} \max_{\mathbf{x} \in T} c(\mathbf{x}) \\ \max_{\mathbf{x} \in X} f(\mathbf{x}) \end{array} \right\} \geq \max_{\mathbf{x} \in X} c(\mathbf{x})
$$

- T: candidate solutions;
- $X \subseteq T$ feasible solutions;
- $f(\mathbf{x}) \geq c(\mathbf{x})$

Relaxations

How to construct relaxations?

1. $IP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P \cap \mathbb{Z}^n\}, P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$
   $LP : \max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in P\}$
   Better formulations give better bounds ($P_1 \subseteq P_2$)

   Proposition

   (i) *If a relaxation LP is infeasible, the original problem IP is infeasible.*

   (ii) *Let $\mathbf{x}^*$ be optimal solution for LP. If $\mathbf{x}^* \in X$ and $f(\mathbf{x}^*) = c(\mathbf{x}^*)$ then $\mathbf{x}^*$ is optimal for IP.*

2. Combinatorial relaxations to easy problems that can be solved rapidly
   Eg: TSP to Assignment problem Eg: Symmetric TSP to 1-tree

3. Lagrangian relaxation

$$IP: \quad z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in X \subseteq \mathbb{Z}^n\}$$

$$LR: \quad z(\mathbf{u}) = \max\{\mathbf{c}^T\mathbf{x} + \mathbf{u}(\mathbf{b} - A\mathbf{x}) : \mathbf{x} \in X\}$$

$$z(\mathbf{u}) \geq z \qquad \forall \mathbf{u} \geq \mathbf{0}$$

4. Duality:

Definition

Two problems:

$$z = \max\{c(\mathbf{x}) : \mathbf{x} \in X\} \qquad w = \min\{w(\mathbf{u}) : \mathbf{u} \in U\}$$

form a weak-dual pair if $c(\mathbf{x}) \leq w(\mathbf{u})$ for all $\mathbf{x} \in X$ and all $\mathbf{u} \in U$.
When $z = w$ they form a strong-dual pair

Proposition

$z = \max\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \le \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+\}$ *and* $w^{LP} = \min\{\mathbf{u}^T\mathbf{b} : A^T\mathbf{u} \ge \mathbf{c}, \mathbf{u} \in \mathbb{R}^m_+\}$ *(ie, dual of linear relaxation) form a weak-dual pair.*

Proposition

Let IP and D be weak-dual pair:

(i) *If D is unbounded, then IP is infeasible*

(ii) *If* $\mathbf{x}^* \in X$ *and* $\mathbf{u}^* \in U$ *satisfy* $c(\mathbf{x}^*) = w(\mathbf{u}^*)$ *then* $\mathbf{x}^*$ *is optimal for IP and* $\mathbf{u}^*$ *is optimal for D.*

The advantage is that we do not need to solve an LP like in the LP relaxation to have a bound, any feasible dual solution gives a bound.

Examples

Weak pairs:
Matching: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^m\}$
V. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T y \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^n\}$

Proof: consider LP relaxations, then $z \leq z^{LP} = w^{LP} \leq w$.
(strong when graphs are bipartite)

Weak pairs:
S. Packing: $z = \max\{\mathbf{1}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^n\}$
S. Covering: $w = \min\{\mathbf{1}^T\mathbf{y} : A^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \in \mathbb{Z}_+^m\}$