DM545/DM871

Linear and Integer Programming

Lecture 13

Network Flows, Cntd

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Outline

$$
\begin{array}{r|cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
i & 1 & . & \cdots & -1 & \cdots & . & = & b_i \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
n & . & . & \cdots & . & \cdots & . & = & b_n \\
\hline
e_1 & 1 & & & & & & \leq & u_1 \\
e_2 & & 1 & & & & & \leq & u_2 \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
(i,j) & & & & 1 & & & \leq & u_{ij} \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
e_m & & & & & & 1 & \leq & u_m
\end{array}
$$

Outline

1. Duality in Network Flow Problems

2. Network Simplex

Shortest Path - Dual LP

$$z = \min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = 1 \qquad \text{for } i = s \qquad (\pi_s)$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = 0 \qquad \forall i \in V \setminus \{s, t\} \qquad (\pi_i)$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = -1 \qquad \text{for } i = t \qquad (\pi_t)$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

Dual problem:

$$g^{LP} = \max \pi_s - \pi_t$$

$$\pi_j - \pi_i \leq c_{ij} \qquad \forall ij \in A$$

Hence, the shortest path can be found by potential values π_i on nodes such that $\pi_s = z, \pi_t = 0$ and $\pi_j - \pi_i \leq c_{ij}$ for $ij \in A$

Maximum (s, t)-Flow

Adding a backward arc from t to s:

$$
\begin{aligned}
z = \max x_{ts} & & & \\
\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = 0 & & \forall i \in V & & (\pi_i) \\
x_{ij} \leq u_{ij} & & \forall ij \in A & & (w_{ij}) \\
x_{ij} \geq 0 & & \forall ij \in A & &
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
g^{LP} = \min \sum_{ij \in A} u_{ij} w_{ij} & & \\
\pi_i - \pi_j + w_{ij} \geq 0 & & \forall ij \in A \\
\pi_t - \pi_s \geq 1 & & \\
w_{ij} \geq 0 & & \forall ij \in A
\end{aligned}
$$

$$
\begin{array}{r|cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
i & 1 & . & \cdots & -1 & \cdots & . & = & b_i \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
n & . & . & \cdots & . & \cdots & . & = & b_n \\
\hline
e_1 & 1 & & & & & & \leq & u_1 \\
e_2 & & 1 & & & & & \leq & u_2 \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
(i,j) & & & & 1 & & & \leq & u_{ij} \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
e_m & & & & & & 1 & \leq & u_m
\end{array}
$$

$$
\begin{aligned}
g^{LP} = \min \sum_{ij \in A} u_{ij} w_{ij} & & (1)\\
\pi_i - \pi_j + w_{ij} \geq 0 & \qquad \forall ij \in A & (2)\\
\pi_t - \pi_s \geq 1 & & (3)\\
w_{ij} \geq 0 & \qquad \forall ij \in A & (4)
\end{aligned}
$$

- Without (3) all potentials would go to 0.
- Keep w low because of objective function
- Keep all potentials low $\rightsquigarrow$ (3) $\pi_s = 0, \pi_t = 1$
- Cut C: on left =1 on right =0. Where is the transition?
- Vars w identify the cut $\rightsquigarrow$ $\pi_j - \pi_i + w_{ij} \geq 0 \rightsquigarrow w_{ij} = 1$

  $$w_{ij} = \begin{cases} 1 & if \;\; ij \in C \\ 0 & otherwise \end{cases}$$

  for those arcs that minimize the cut capacity $\sum_{ij \in A} u_{ij} w_{ij}$
- Complementary slackness: $w_{ij} = 1 \implies x_{ij} = u_{ij}$

Theorem

A strong dual to the max (st)-flow is the minimum (st)-cut problem:

$$\min_{X} \left\{ \sum_{ij \in A : i \in X, j \notin X} u_{ij} : s \in X \subset V \setminus \{t\} \right\}$$

Max Flow Algorithms

Optimality Condition

- Ford Fulkerson augmenting path algorithm $O(m|x^*|)$
- Edmonds-Karp algorithm (augment by shortest path) in $O(nm^2)$
- Dinic algorithm in layered networks $O(n^2m)$
- Karzanov's push relabel $O(n^2m)$

Min Cost Flow - Dual LP

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = b_i \qquad \forall i \in V \qquad (\pi_i)$$

$$x_{ij} \leq u_{ij} \qquad \forall ij \in A \qquad (w_{ij})$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

Dual problem:

$$\max \sum_{i \in V} b_i \pi_i - \sum_{ij \in E} u_{ij} w_{ij} \tag{1}$$

$$-c_{ij} - \pi_i + \pi_j \leq w_{ij} \qquad \forall ij \in E \tag{2}$$

$$w_{ij} \geq 0 \qquad \forall ij \in A \tag{3}$$

- define reduced costs $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$, hence (2) becomes $-\bar{c}_{ij} \leq w_{ij}$
- $u_e = \infty$ then $w_e = 0$ (from obj. func) and $\bar{c}_{ij} \geq 0$ (optimality condition)
- $u_e < \infty$ then $w_e \geq 0$ and $w_e \geq -\bar{c}_{ij}$ then $w_e = \max\{0, -\bar{c}_{ij}\}$, hence w_e is determined by others and irrelevant
- Complementary slackness th. for optimal solutions:
  each primal variable $\times$ the corresponding dual slack must be equal 0, ie, $x_e(\bar{c}_e + w_e) = 0$;
  - $x_e > 0$ then $-\bar{c}_e = w_e = \max\{0, -\bar{c}_e\}$,
    $x_e > 0 \implies -\bar{c}_e \geq 0$ or equivalently (by negation) $\bar{c}_e > 0 \implies x_e = 0$

  each dual variable $\times$ the corresponding primal slack must be equal 0, ie, $w_e(x_e - u_e) = 0$;
  - $w_e > 0$ then $x_e = u_e$
    $-\bar{c}_e > 0 \implies x_e = u_e$ or equivalently $\bar{c}_e < 0 \implies x_e = u_e$

  Hence:
  $\bar{c}_e > 0$ then $x_e = 0$
  $\bar{c}_e < 0$ then $x_e = u_e \neq \infty$

Min Cost Flow Algorithms

The conditions derived can be used to define a solution approach for the minimum cost flow problem.

Note that if a set of potentials $\pi_i, i \in V$ are given, and the cost of a circuit wrt. the reduced costs for the edges ($\bar{c}_{ij} = c_{ij} + \pi_j - \pi_j$) are calculated, the cost remains the same as the original costs as the potentials are "telescoped" to 0.

Theorem (Optimality conditions)

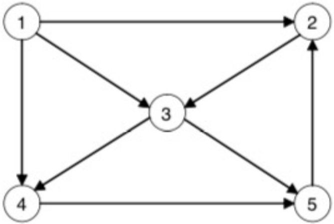
Let $\mathbf{x}$ be feasible flow in $N(V, A, \mathbf{l}, \mathbf{u}, \mathbf{b})$ then $\mathbf{x}$ is min cost flow in N iff $N(\mathbf{x})$ contains no directed cycle of negative cost.

Note that a (directed) circuit with negative cost in $N(x)$ corresponds to a negative cost cycle in N, if costs are added for forward edges and subtracted for backward edges.

- Cycle canceling algorithm with Bellman Ford Moore for negative cycles $O(nm^2UC)$, $U = \max|u_e|$, $C = \max|c_e|$
- Build up algorithms $O(n^2mM)$, $M = \max|b(v)|$

Outline

Min Cost Flow



$$A = \begin{bmatrix} 1 & 1 & 1 & & & & & \\ -1 & & & 1 & & & & -1 \\ & -1 & & -1 & 1 & 1 & & \\ & & -1 & & -1 & & 1 & \\ & & & & & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{array}{lrrrrrrrrcr} \min & \multicolumn{8}{l}{10x_{12} + 8x_{13} + x_{14} + 2x_{23} + x_{34} + 4x_{35} + 12x_{45} - 7x_{52}} & & \\ & x_{12} & +x_{13} & +x_{14} & & & & & & = & 10 \\ & -x_{12} & & & +x_{23} & & & & -x_{52} & = & 4 \\ & & -x_{13} & & -x_{23} & +x_{34} & +x_{35} & & & = & 0 \\ & & & -x_{14} & & -x_{34} & & +x_{45} & & = & -6 \\ & & & & & & -x_{35} & -x_{45} & +x_{52} & = & -8 \\ & x \geq 0 & & & & & & & & & \end{array}$$

- A is not full-rank: adding all rows $\rightsquigarrow$ null vector, i.e., the rows of A are not linearly indep.

- Since we assume that total supply equal total demand, i.e., $\sum_{i \in V} b_i = 0$ then $\text{rank}[A] = \text{rank}[A\ \mathbf{b}]$.

- Hence, one of the equations can be canceled.

- assume network N is connected
- cycle: here, a set of arcs forming a closed path (i.e., a path in which the first and the last node of the path coincide) when ignoring their orientation
- spanning tree: here, a tree that reaches everynode (it coincides with the classical notion of spanning tree if one disregards arc orientation).

Theorem (Spanning Trees)

For an undirected graph $D' = (N, A')$, the following are equivalent:
(a) G' is a tree (acyclic and connected);
(b) G' is acyclic and has $n - 1$ arcs; and
(c) G' is connected and has $n - 1$ arcs.

Since we know that the matrix A is not full-rank, a basis of A consists of only $n-1$ linearly independent columns of A. These columns correspond to a collection of arcs of the flow network.

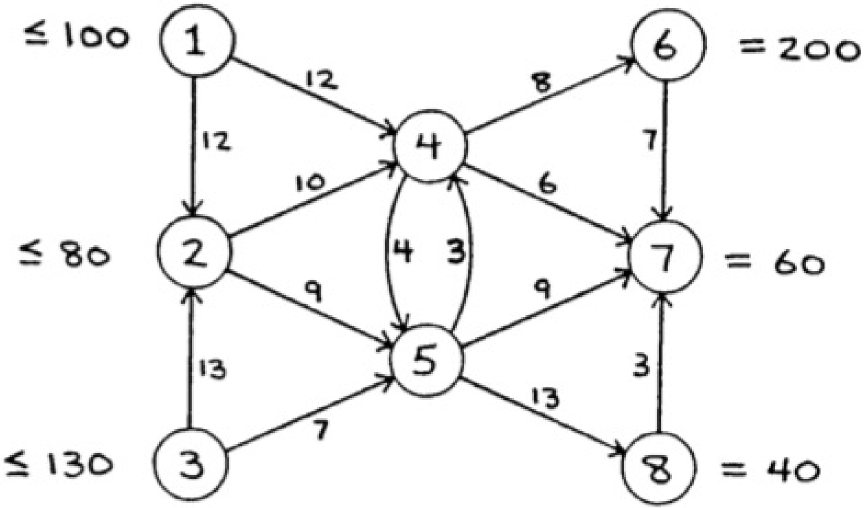
Theorem

Given a connected flow network, letting A be its incidence matrix, a submatrix B of size $(n-1)\times(n-1)$ is a basis of A ***if and only if*** *the arcs associated with the columns of B form a spanning tree.*

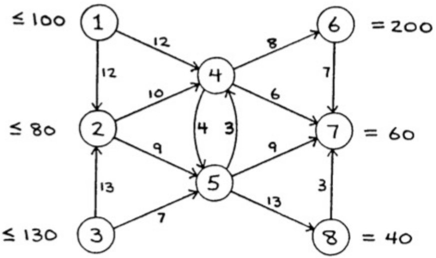
Proof:
if columns from A correspond to a spanning tree $\implies$ they are lin. indep., B is upper triangular
if a subset of columns of A are a basis $\implies$ they are $n-1$ and acyclic

Hence, all basic feasible solutions explored by the simplex algorithm are spanning trees of the flow network.
As for any LP, also in min-cost flow problems there are feasible, infeasible and degenerate bases. (feasible if $\mathbf{x}_B = A_B^{-1}\mathbf{b} \geq \mathbf{0}$).

Example



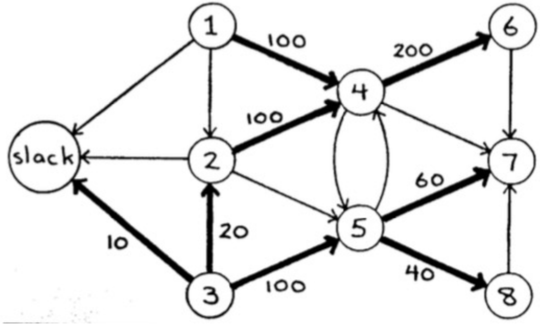
Example



$$12x_{12} + 12x_{14} + 10x_{24} + 9x_{25} + 13x_{32} + 7x_{35} + 4x_{45} + 8x_{46} + 6x_{47} + 3x_{54} + 9x_{57} + 13x_{58} + 7x_{67} + 3x_{87}$$

$$\begin{aligned}
+x_{12} + x_{14} + s_1 &= 100\\
-x_{12} + x_{24} + x_{25} - x_{32} + s_2 &= 80\\
+ x_{32} + x_{35} + s_3 &= 130\\
- x_{14} - x_{24} + x_{45} + x_{46} + x_{47} - x_{54} &= 0\\
- x_{25} - x_{35} - x_{45} + x_{54} + x_{57} + x_{58} &= 0\\
- x_{46} + x_{67} &= -200\\
- x_{47} - x_{57} - x_{67} - x_{87} &= -60\\
- x_{58} + x_{87} &= -40\\
x_{12}, x_{14}, x_{24}, x_{25}, x_{32}, x_{35}, x_{45}, x_{46}, x_{47}, x_{54}, x_{57}, x_{58}, x_{67}, x_{87}, s_1, s_2, s_3 &\geq 0
\end{aligned}$$

Example



$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

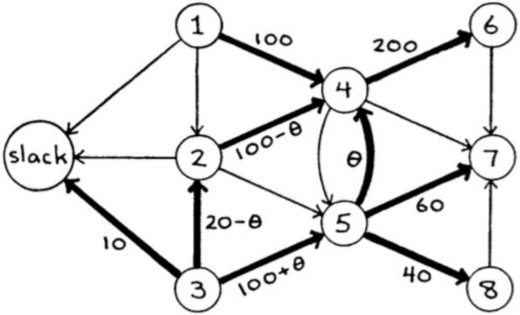
- solve $B\mathbf{x}_B = \mathbf{b}$ in value of variables to check feasibility; easy because of structure or because done by updates.
- solve $\boldsymbol{\pi}^T B = \mathbf{c}_B^T$ in $\boldsymbol{\pi}$ (dual potential variables to derive reduced costs); easy because of structure of B.
- calculate $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$

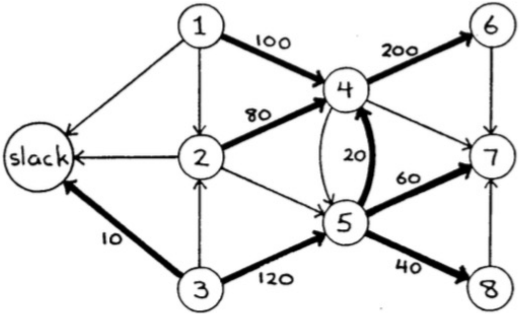
$$\begin{aligned}
\pi_1 - \pi_4 &= 12\\
\pi_2 - \pi_4 &= 10\\
\pi_3 - \pi_2 &= 13\\
\pi_3 - \pi_5 &= 7\\
\pi_4 - \pi_6 &= 8\\
\pi_5 - \pi_7 &= 9\\
\pi_5 - \pi_8 &= 13\\
\pi_3 &= 0
\end{aligned}$$

$$\begin{array}{lclcl}
\pi_3 = 0 & \text{and} & \pi_3 - \pi_5 = 7 & \Rightarrow & \pi_5 = -7\\
\pi_5 = -7 & \text{and} & \pi_5 - \pi_8 = 13 & \Rightarrow & \pi_8 = -20\\
\pi_5 = -7 & \text{and} & \pi_5 - \pi_7 = 9 & \Rightarrow & \pi_7 = -16\\
\pi_3 = 0 & \text{and} & \pi_3 - \pi_2 = 13 & \Rightarrow & \pi_2 = -13\\
\pi_2 = -13 & \text{and} & \pi_2 - \pi_4 = 10 & \Rightarrow & \pi_4 = -23\\
\pi_4 = -23 & \text{and} & \pi_4 - \pi_6 = 8 & \Rightarrow & \pi_6 = -31\\
\pi_4 = -23 & \text{and} & \pi_1 - \pi_4 = 12 & \Rightarrow & \pi_1 = -11
\end{array}$$

$$\begin{aligned}
d_{12} &= c_{12} - \pi_1 + \pi_2 = 12 - (-11) + (-13) = 10\\
d_{25} &= c_{25} - \pi_2 + \pi_5 = 9 - (-13) + (-7) = 15\\
d_{45} &= c_{45} - \pi_4 + \pi_5 = 4 - (-23) + (-7) = 20\\
d_{54} &= c_{54} - \pi_5 + \pi_4 = 3 - (-7) + (-23) = -13\\
d_{47} &= c_{47} - \pi_4 + \pi_7 = 6 - (-23) + (-16) = 13\\
d_{67} &= c_{67} - \pi_6 + \pi_7 = 7 - (-31) + (-16) = 22\\
d_{87} &= c_{87} - \pi_8 + \pi_7 = 3 - (-20) + (-16) = 7
\end{aligned}$$

$$\begin{aligned}
d_1 &= 0 - \pi_1 = -(-11) = 11\\
d_2 &= 0 - \pi_2 = -(-13) = 13
\end{aligned}$$





How much can we increase the flow θ through (54)?
Until (32) reaches zero

- It can be proved that, because the basis corresponds to a tree, the equations can always be solved by simple substitution.

- The order of substitution can always be found by "walking around the tree".

- Efficient implementations further reduce the cost of determining π by updating it as they walk around the tree, rather than computing it anew at each iteration.

- When the network simplex steps are to be carried out by a computer, it is not so obvious how

- A few concise and clever data structures are used to represent the basis tree in a way that allows the walk around the tree and finding the circuit induced by the entering arc efficiently.

- The data structures can themselves be efficiently updated as the tree changes from iteration to iteration.