Outline

1. Vehicle Routing

2. Integer Programming

Problem Definition

Vehicle Routing: distribution of goods between depots and customers.
Delivery, collection, transportation.
Examples: solid waste collection, street cleaning, school bus routing, dial-a-ride systems, transportation of handicapped persons, routing of salespeople and maintenance unit.

Vehicle Routing Problems

**Input:** Vehicles, depots, road network, costs and customers requirements.

**Output:** Set of routes such that:
- requirement of customers are fulfilled,
- operational constraints are satisfied and
- a global transportation cost is minimized.
Refinement

Road Network
- represented by a (directed or undirected) complete graph
- travel costs and travel times on the arcs obtained by shortest paths

Customers
- vertices of the graph
- collection or delivery demands
- time windows for service
- service time
- subset of vehicles that can serve them
- priority (if not obligatory visit)

Vehicles
- capacity
- types of goods
- subsets of arcs traversable
- fix costs associated to the use of a vehicle
- distance dependent costs
- a-priori partition of customers
- home depot in multi-depot systems
- drivers with union contracts

Operational Constraints
- vehicle capacity
- delivery or collection
- time windows
- working periods of the vehicle drivers
- precedence constraints on the customers

Objectives
- minimization of global transportation cost (variable + fixed costs)
- minimization of the number of vehicles
- balancing of the routes
- minimization of penalties for un-served customers

History:
Dantzig, Ramser “The truck dispatching problem”, Management Science, 1959
Clark, Wright, “Scheduling of vehicles from a central depot to a number of delivery points”. Operation Research. 1964
Vehicle Routing Problems

- Capacitated (and Distance Constrained) VRP (CVRP and DCVRP)
- VRP with Time Windows (VRPTW)
- VRP with Backhauls (VRPB)
- VRP with Pickup and Delivery (VRPPD)
- Periodic VRP (PVRP)
- Multiple Depot VRP (MDVRP)
- Split Delivery VRP (SDVRP)
- VRP with Satellite Facilities (VRPSF)
- Site Dependent VRP
- Open VRP
- Stochastic VRP (SVRP)
- ...

Task:
Find collection of $K$ circuits with minimum cost, defined as the sum of the costs of the arcs of the circuits and such that:
- each circuit visits the depot vertex
- each customer vertex is visited by exactly one circuit; and
- the sum of the demands of the vertices visited by a circuit does not exceed the vehicle capacity $Q$.

Note: lower bound on $K$
- $\lceil d(V')/Q \rceil$
- number of bins in the associated Bin Packing Problem

Capacitated Vehicle Routing (CVRP)

Input: (common to all VRPs)
- (di)graph (strongly connected, typically complete) $G(V, A)$, where $V = \{0, \ldots, n\}$ is a vertex set:
  - $0$ is the depot.
  - $V' = V \setminus \{0\}$ is the set of $n$ customers
  - $A = \{(i, j) : i, j \in V\}$ is a set of arcs
- $C$ a matrix of non-negative costs or distances $c_{ij}$ between customers $i$ and $j$ (shortest path or Euclidean distance)
  - $c_{ik} + c_{kj} \geq c_{ij} \forall i, j \in V$
- a non-negative vector of customer demands $d_i$
- a set of $K$ (identical!) vehicles with capacity $Q, d_i \leq Q$

A feasible solution is composed of:
- a partition $R_1, \ldots, R_m$ of $V$;
- a permutation $\pi_i$ of $R_i \cup \{0\}$ specifying the order of the customers on route $i$.

A route $R_i$ is feasible if $\sum_{i=\pi_1}^{\pi_m} d_i \leq Q$.

The cost of a given route ($R_i$) is given by: $F(R_i) = \sum_{i=\pi_1}^{\pi_m} c_{i,i+1}$

The cost of the problem solution is: $F_{VRP} = \sum_{i=1}^{m} F(R_i)$. 
Relation with TSP
- VRP with $K = 1$, no limits, no (any) depot, customers with no demand $\Rightarrow$ TSP
- VRP is a generalization of the Traveling Salesman Problem (TSP) $\Rightarrow$ is NP-Hard.
- VRP with a depot, $K$ vehicles with no limits, customers with no demand $\Rightarrow$ Multiple TSP = one origin and $K$ salesman
- Multiple TSP is transformable in a TSP by adding $K$ identical copies of the origin and making costs between copies infinite.

Vehicle Routing with Time Windows (VRPTW)

Further Input:
- each vertex is also associated with a time interval $[a_i, b_j]$.
- each arc is associated with a travel time $t_{ij}$
- each vertex is associated with a service time $s_i$

Task:
Find a collection of $K$ simple circuits with minimum cost, such that:
- each circuit visit the depot vertex
- each customer vertex is visited by exactly one circuit; and
- the sum of the demands of the vertices visited by a circuit does not exceed the vehicle capacity $Q$.
- for each customer $i$, the service starts within the time windows $[a_i, b_i]$ (it is allowed to wait until $a_i$ if early arrive)

Variants of CVRP:
- minimize number of vehicles
- different vehicles $Q_k$, $k = 1, \ldots, K$
- Distance-Constrained VRP: length $t_{ij}$ on arcs and total duration of a route cannot exceed $T$ associated with each vehicle
generally $c_{ij} = t_{ij}$
(service times $s_i$ can be added to the travel times of the arcs: $t_{ij}' = t_{ij} + s_i/2 + s_j/2$)
- Distance constrained CVRP

Time windows induce an orientation of the routes.
Variants

- Minimize number of routes
- Minimize hierarchical objective function
- Makespan VRP with Time Windows (MPTW) minimizing the completion time
- Delivery Man Problem with Time Windows (DMPTW) minimizing the sum of customers waiting times

Solution Techniques for CVRP

- Integer Programming
- Construction Heuristics
- Local Search
- Metaheuristics
- Hybridization with Constraint Programming

Outline

1. Vehicle Routing
2. Integer Programming

Basic Models

- vehicle flow formulation
  - integer variables on the edges counting the number of time it is traversed
  - two or three index variables

- commodity flow formulation
  - additional integer variables representing the flow of commodities along the paths traveled by the vehicles

- set partitioning formulation