Elastic Collisions

Algebraic Derivation of Post-Collision Velocities (1D)

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TeX Source: http://www.rutski89.com/static/ec.tex

Our goal is to find the new velocities (v_1 and v_2) of two masses after they have collided in 1D space with full elasticity. We start by noting that the very definition of a "fully elastic" collision is one in which the sums of the objects' momentums and kinetic energies remain unchanged after impact:

$$\left(\frac{m_1 o_1^2}{2} + \frac{m_2 o_2^2}{2}\right) = \left(\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}\right)$$

and

$$(m_1 o_1 + m_2 o_2) = (m_1 v_1 + m_2 v_2)$$

o_1 and o_2 in the above denote the *original* velocities, as before impact.

We can find v_1 by solving these simultaneous equations algebraically. But first, it would do us good to establish a notation without those messy sub-scripts:

$$\begin{aligned} m_1 &= s \\ m_2 &= t \\ o_1 &= x \\ o_2 &= y \\ v_1 &= w \\ v_2 &= z \end{aligned}$$

So then, to restate for the new notation: we are trying to find w given

$$\left(\frac{sx^2}{2} + \frac{ty^2}{2}\right) = \left(\frac{sw^2}{2} + \frac{tz^2}{2}\right)$$

and

$$(sx + ty) = (sw + tz)$$

Our first problem is that we have two unknowns, w and z. This won't do, we want w to be the single unknown; we must eliminate z. This can be done by stating both equations in terms of z^2, and then setting them equal to each other:

firstly, kinetic energy

$$\begin{aligned} \left(\frac{sx^2}{2} + \frac{ty^2}{2}\right) &= \left(\frac{sw^2}{2} + \frac{tz^2}{2}\right) \\ \left(sx^2 + ty^2\right) &= \left(sw^2 + tz^2\right) \\ \left(sx^2 + ty^2\right) - sw^2 &= tz^2 \\ \frac{\left(sx^2 + ty^2\right) - sw^2}{t} &= z^2 \end{aligned}$$

and secondly, momentum

$$\begin{aligned} (sx + ty) &= sw + tz \\ (sx + ty) - sw &= tz \\ \frac{(sx + ty) - sw}{t} &= z \\ \left(\frac{(sx + ty) - sw}{t}\right)^2 &= z^2 \end{aligned}$$

Setting these two equal to each other now gives

$$\left(\frac{sx+ty-sw}{t}\right)^2 = \frac{sx^2+ty^2-sw^2}{t}$$

which has the desired attribute of having only a single unknown, w. Take note that we have both w and w^2 terms here. The next logical step is thus to rearrange to the quadratic form $aw^2+bw+c=0$.

Let's first expand $\left(\frac{sx+ty-sw}{t}\right)^2$. We quickly see that it's $\frac{(sx+ty-sw)^2}{t^2}$, so we can just expand $(sx+ty-sw)^2$ on its own:

$$\begin{gathered}
(sx+ty-sw)^2 \\
(sx+ty-sw)(sx+ty-sw) \\
sx(sx+ty-sw)+ty(sx+ty-sw)-sw(sx+ty-sw) \\
(s^2x^2+stxy-s^2xw)+(stxy+t^2y^2-styw)+(-s^2xw-styw+s^2w^2) \\
(s^2x^2+t^2y^2+s^2w^2)+(stxy+stxy)+(-s^2xw-s^2xw)+(-styw-styw) \\
(s^2x^2+t^2y^2+s^2w^2)+(2stxy-2s^2xw-2styw)
\end{gathered}$$

and so we have

$$\left(\frac{sx+ty-sw}{t}\right)^2 = \frac{(s^2x^2+t^2y^2+s^2w^2)+(2stxy-2s^2xw-2styw)}{t^2}$$

and we can now replace the $\left(\frac{sx+ty-sw}{t}\right)^2$ with our new definition

$$\begin{aligned}
\left(\frac{sx+ty-sw}{t}\right)^2 &= \frac{sx^2+ty^2-sw^2}{t} \\
\frac{(s^2x^2+t^2y^2+s^2w^2)+(2stxy-2s^2xw-2styw)}{t^2} &= \frac{sx^2+ty^2-sw^2}{t}
\end{aligned}$$

Next, let's multiply both sides by t^2 (to get rid of the big horizontal lines)

$$(s^2x^2+t^2y^2+s^2w^2)+(2stxy-2s^2xw-2styw) = stx^2+t^2y^2-stw^2$$

now drop the t^2y^2 from both sides

$$\begin{gathered}
(s^2x^2+\mathbf{t^2y^2}+s^2w^2)+(2stxy-2s^2xw-2styw) = stx^2+\mathbf{t^2y^2}-stw^2 \\
(s^2x^2+s^2w^2)+(2stxy-2s^2xw-2styw) = stx^2-stw^2
\end{gathered}$$

then set equal to 0

$$\begin{aligned}
(s^2x^2+s^2w^2)+(2stxy-2s^2xw-2styw) &= (\mathbf{stx^2}-\mathbf{stw^2}) \\
(s^2x^2+s^2w^2)+(2stxy-2s^2xw-2styw)-(\mathbf{stx^2}-\mathbf{stw^2}) &= 0
\end{aligned}$$

then get rid of the parens, taking note that $-(stx^2-stw^2)$ will become $-stx^2+stw^2$

$$s^2x^2+s^2w^2+2stxy-2s^2xw-2styw-stx^2+stw^2=0$$

and finally, rearrange into the $aw^2+bw+c=0$ quadratic form

$$w^2(s^2+st)+w(-2s^2x-2sty)+(s^2x^2+2stxy-stx^2)=0$$

and thus we have

$$
\begin{aligned}
a &= (s^2 + st) \\
b &= (-2s^2x - 2sty) \\
c &= (s^2x^2 + 2stxy - stx^2)
\end{aligned}
$$

To find w (a.k.a v_1), we can now simply use $w = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$; though the algebra, as usual, will be a bit cumbersome.

Let's find $-b$, b^2, $4ac$, and $b^2 - 4ac$ in steps on their own.

$$-b = -(-2s^2x - 2sty) = 2s^2x + 2sty$$

At least that one was easy. b^2 is slightly more involved

$$
\begin{aligned}
&b^2 \\
&\quad = (-2s^2x - 2sty)^2 \\
&\quad = (-2s^2x - 2sty)(-2s^2x - 2sty) \\
&\quad = -2s^2x(-2s^2x - 2sty) - 2sty(-2s^2x - 2sty) \\
&\quad = (4s^4x^2 + \mathbf{4s^3txy}) + (\mathbf{4s^3txy} + 4s^2t^2y^2) \\
&\quad = 4s^4x^2 + \mathbf{8s^3txy} + 4s^2t^2y^2
\end{aligned}
$$

Now for $4ac$

$$
\begin{aligned}
&4ac \\
&\quad = 4(s^2 + st)(s^2x^2 + 2stxy - stx^2) \\
&\quad = 4(s^2(s^2x^2 + 2stxy - stx^2) + st(s^2x^2 + 2stxy - stx^2)) \\
&\quad = 4(s^4x^2 + 2s^3txy - s^3tx^2) + (s^3tx^2 + 2s^2t^2xy - s^2t^2x^2) \\
&\quad = 4s^4x^2 + 8s^3txy - \mathbf{4s^3tx^2} + \mathbf{4s^3tx^2} + 8s^2t^2xy - 4s^2t^2x^2 \\
&\quad = 4s^4x^2 + 8s^3txy + \mathbf{0} + 8s^2t^2xy - 4s^2t^2x^2 \\
&\quad = 4s^4x^2 + 8s^3txy + 8s^2t^2xy - 4s^2t^2x^2
\end{aligned}
$$

Lastly, the rather large $b^2 - 4ac$, which (**thank goodness!**), actually reduces to something very small.

$$
\begin{aligned}
&b^2 - 4ac \\
&\quad = (4s^4x^2 + 8s^3txy + 4s^2t^2y^2) - (4s^4x^2 + 8s^3txy + 8s^2t^2xy - 4s^2t^2x^2) \\
&\quad = (\mathbf{4s^4x^2} - \mathbf{4s^4x^2}) + (\mathbf{8s^3txy} - \mathbf{8s^3txy}) + 4s^2t^2y^2 - 8s^2t^2xy + 4s^2t^2x^2 \\
&\quad = (\mathbf{0}) + (\mathbf{0}) + 4s^2t^2y^2 - 8s^2t^2xy + 4s^2t^2x^2 \\
&\quad = 4\mathbf{s^2t^2}y^2 - 8\mathbf{s^2t^2}xy + 4\mathbf{s^2t^2}x^2 \\
&\quad = s^2t^2(4y^2 - 8xy + 4x^2) \\
&\quad = s^2t^2(2y - 2x)^2
\end{aligned}
$$

The quadratic formula can now be tackled in proper

$$
\begin{aligned}
w & \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{(2s^2x + 2sty) \pm \sqrt{s^2t^2(2y - 2x)^2}}{2(s^2 + st)} \\
&= \frac{(2s^2x + 2sty) \pm (\mathbf{st}(\mathbf{2y} - \mathbf{2x}))}{2(s^2 + st)} \\
&= \frac{(2s^2x + 2sty) \pm (st(2y - 2x))}{\mathbf{2}(\mathbf{s}(\mathbf{s} + \mathbf{t}))} \\
&= \frac{(2s^2x + 2sty) \pm (st(2y - 2x))}{\mathbf{2s}(\mathbf{s} + \mathbf{t})} \\
&= \frac{(2s^2x + 2sty) + (st(2y - 2x))}{2s(s + t)} \\
&= \frac{(2s^2x + 2sty) + \mathbf{2st}(\mathbf{y} - \mathbf{x})}{2s(s + t)} \\
&= \frac{(sx + ty) + t(y - x)}{(s + t)}
\end{aligned}
$$

So there we at long last have it! Our final solution

$$
w = \frac{(sx + ty) + t(y - x)}{(s + t)}
$$

or

$$
v_1 = \frac{(m_1o_1 + m_2o_2) + m_2(o_2 - o_1)}{(m_1 + m_2)}
$$

v_2 is then but a simple mirror image

$$
v_2 = \frac{(m_2o_2 + m_1o_1) + m_1(o_1 - o_2)}{(m_2 + m_1)}
$$