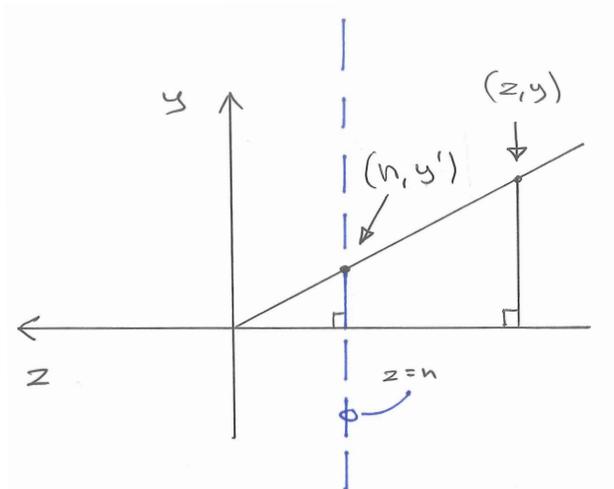


Perspective Projection I

In this note, names denoting vector values are shown in bold (instead of using arrows above the name).

Recall that by perspective projection, we mean moving a point $\mathbf{p} = (x, y, z) \in \mathbb{R}^3$ to the point of intersection between the *viewing plane* and the line going through the camera position and \mathbf{p} . In 3D graphics, one usually takes origo to be the camera position and the viewing plane to be a plane perpendicular to the z -axis. Such a plane is given by the equation $z = n$, where n is some non-zero value chosen by the user. In OpenGL, n is usually negative. In other words, the camera is positioned in origo and is looking down the negative z -axis. Note that such an intersection is defined if and only if \mathbf{p} does not lie in the xy -plane. We in the following assume that $\mathbf{p} = (x, y, z)$ fulfills $z < 0$, unless otherwise noted.

The following figure shows the situation from the side (y -axis is up).



Using the fact that triangles with same angles are scalings of each other, it follows from the figure that $n = cz$ and $y' = cy$ for some scaling factor c .

From this, we get $n/z = c$ and then $y' = yn/z$. Thus, the projected y -value is $y' = yn/z$. A similar figure can be drawn with the y -axis exchanged by the x -axis (which means looking from below), from which the same argument shows that the projected x -value is $x' = xn/z$. In short, the perspective projection is given by the following mapping \mathbf{f} :

$$\mathbf{f}(x, y, z) = \begin{pmatrix} xn/z \\ yn/z \end{pmatrix}$$

We now prove the following fact. It is interesting by itself, and e.g. implies that triangles are projected to triangles, such that we can just project the three vertices of a triangle, and then find the full projected triangle by drawing lines between the three projected points and then filling out the area (rasterization). We furthermore will have good use of the function λ constructed during the proof.¹

Theorem 1 *For the mapping \mathbf{f} , line segments not crossing the xy -plane are mapped to line segments.*

Proof: Let \mathbf{l} be a given line segment from $\mathbf{p}_0 = (x_0, y_0, z_0)$ to $\mathbf{p}_1 = (x_1, y_1, z_1)$, where z_0 and z_1 are non-zero and have the same sign. The points on \mathbf{l} can be expressed as

$$\mathbf{l}(s) = \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0) = \begin{pmatrix} x_0 + s(x_1 - x_0) \\ y_0 + s(y_1 - y_0) \\ z_0 + s(z_1 - z_0) \end{pmatrix}$$

for $s \in [0, 1]$. Let us define

$$\lambda(s) = \frac{z_1 s}{z_0 + (z_1 - z_0)s}.$$

¹In the handwritten notes *Perspective projection*, λ is called \tilde{f} .

Looking at $\mathbf{f}(\mathbf{l}(s))$ we can calculate

$$\begin{aligned}\mathbf{f}(\mathbf{l}(s)) &= \begin{pmatrix} \frac{(x_0 + s(x_1 - x_0))n}{z_0 + s(z_1 - z_0)} \\ \frac{(y_0 + s(y_1 - y_0))n}{z_0 + s(z_1 - z_0)} \end{pmatrix} \\ &= \begin{pmatrix} x_0n/z_0 + \lambda(s)(x_1n/z_1 - x_0n/z_0) \\ y_0n/z_0 + \lambda(s)(y_1n/z_1 - y_0n/z_0) \end{pmatrix} \\ &= \mathbf{f}(\mathbf{p}_0) + \lambda(s)(\mathbf{f}(\mathbf{p}_1) - \mathbf{f}(\mathbf{p}_0)),\end{aligned}$$

where the calculations behind the second equality is relegated to Lemma 3 below.

By Lemma 2 below, for $s \in [0, 1]$ the points in

$$\mathbf{f}(\mathbf{p}_0) + \lambda(s)(\mathbf{f}(\mathbf{p}_1) - \mathbf{f}(\mathbf{p}_0))$$

will be the same as the points

$$\mathbf{f}(\mathbf{p}_0) + t(\mathbf{f}(\mathbf{p}_1) - \mathbf{f}(\mathbf{p}_0))$$

for $t \in [0, 1]$, that is, exactly the points of the line segment from $\mathbf{f}(\mathbf{p}_0)$ to $\mathbf{f}(\mathbf{p}_1)$. \square

In short, the proof above shows that when traversing the line

$$\mathbf{l}(s) = \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0), \quad s \in [0, 1],$$

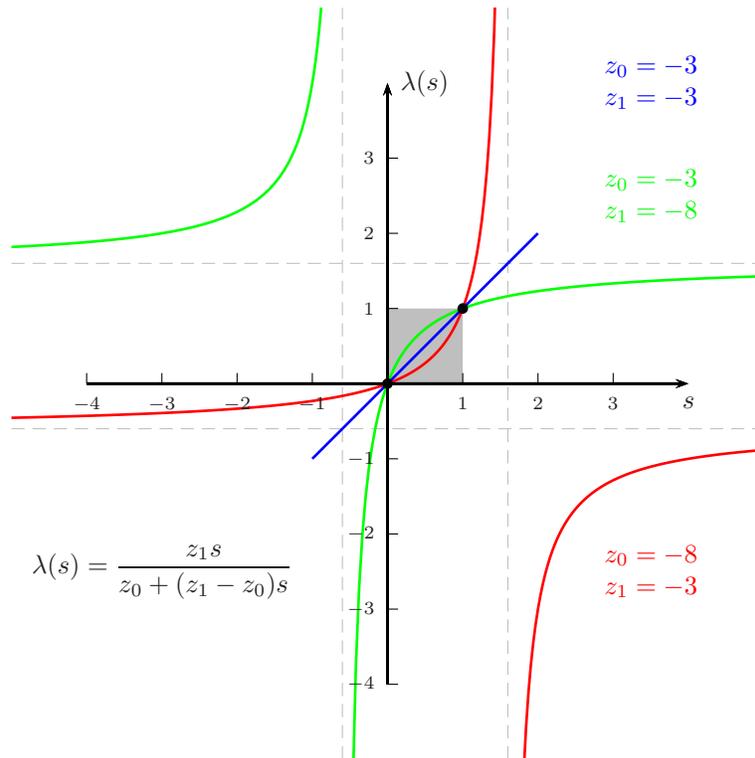
between the endpoints \mathbf{p}_0 and \mathbf{p}_1 , the mapping $\mathbf{f}(\mathbf{l}(s))$ of the traversed point $\mathbf{l}(s)$ will itself traverse the line

$$\mathbf{f}(\mathbf{p}_0) + t(\mathbf{f}(\mathbf{p}_1) - \mathbf{f}(\mathbf{p}_0)), \quad t \in [0, 1]$$

between the mapped endpoints $\mathbf{f}(\mathbf{p}_0)$ and $\mathbf{f}(\mathbf{p}_1)$.

However, the lines are *not* traversed at the same speed in the two expressions above. This is to be expected, as the following figure shows.

Below, we show the function $\lambda(s)$ for three pairs of values of z_0 and z_1 . The gray box illustrates Lemma 2. Asymptotes are also shown. E.g., for $s \rightarrow \infty$ we have $\lambda(s) \rightarrow z_1/(z_1 - z_0)$, which for the green example is $8/5$. That $t = \lambda(s)$ should be bounded for $s \rightarrow \infty$ is clearly to be expected from the figure above.



Lemma 2 *If z_0 and z_1 are non-zero and have the same sign, λ is a bijective (i.e., 1-1 and onto) mapping from $[0, 1]$ to $[0, 1]$.*

Proof: We note that $\lambda(0) = 0$ and $\lambda(1) = 1$. Hence, if we can prove λ continuous and strictly increasing on $[0, 1]$, we are done.

Addition and multiplication are continuous functions. Division is a continuous function on intervals where the divisor is not zero. Hence λ is continuous if the denominator is not zero. Since $z_0 + (z_1 - z_0)s = 0 \Leftrightarrow s = -z_0/(z_1 - z_0) =$

$z_0/(z_0 - z_1)$, we can see that the denominator in $\lambda(s)$ is not zero for any $s \in [0, 1]$ by a case analysis: i) for $z_1 > z_0 > 0$ or $0 > z_0 \geq z_1$ a denominator of zero means $s < 0$, ii) for $z_0 > z_1 > 0$ or $0 > z_1 \geq z_0$ a denominator of zero means $s > 1$, iii) for $z_0 = z_1$ we simply have $\lambda(s) = s$. In all cases, $\lambda(s)$ is continuous on $[0, 1]$.

To show that λ is increasing, we use the quotient rule for differentiation and get $\lambda'(s) = (z_1(z_0 + (z_1 - z_0)s) - z_1s(z_1 - z_0))/(z_0 + (z_1 - z_0)s)^2 = z_1z_0/(z_0 + (z_1 - z_0))^2$, hence $\lambda'(s) > 0$ when z_0 and z_1 are non-zero and have the same sign. \square

Lemma 3 *We have*

$$\frac{(x_0 + s(x_1 - x_0))n}{z_0 + s(z_1 - z_0)} = x_0n/z_0 + \lambda(s)(x_1n/z_1 - x_0n/z_0),$$

and similar with y in x 's place.

Proof: See last page of the handwritten notes *Perspective projection*. There, the substitution $w = z/n$ is used, but the calculations are essentially the same. \square